A Hierarchy of Supermartingales for ω-Regular Verification

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We propose new supermartingale-based certificates for verifying almost sure satisfaction of ω-regular properties: (1) *generalised Streett supermartingales* (GSSMs) and their lexicographic extension (LexGSSMs), (2) *distribution-valued Streett supermartingales* (DVSSMs), and (3) *progress-measure supermartingales* (PMSMs) and their lexicographic extension (LexPMSMs). GSSMs, LexGSSMs, and DVSSMs are derived from least-fixed point characterisations of positive recurrence and null recurrence of Markov chains with respect to given Streett conditions; and PMSMs and LexPMSMs are probabilistic extensions of parity progress measures. We study the hierarchy among these certificates and existing certificates, namely Streett supermartingales, by comparing the classes of problems that can be verified by each type of certificates. Notably, we show that our certificates are strictly more powerful than Streett supermartingales. We also prove completeness of GSSMs for positive recurrence and of DVSSMs for null recurrence: DVSSMs are, in theory, the most powerful certificates in the sense that for any Markov chain that almost surely satisfies a given ω-regular property, there exists a DVSSM certifying it. We provide a sound and relatively complete algorithm for synthesising LexPMSMs, the second most powerful certificates in the hierarchy. We have implemented a prototype tool based on this algorithm, and our experiments show that our tool can successfully synthesise certificates for various examples including those that cannot be certified by existing supermartingales.

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Verification of ω-regular properties of probabilistic systems

Verification of probabilistic programs has been actively studied. *Martingale-based methods* are one of the major approaches in this area. Recently, supermartingales have found applications in verifying *ω-regular properties* of probabilistic programs [1] [2] [7] [5]. For example, consider a one-dimensional fair random walk. One may ask whether the position $x = 0$ is visited infinitely often (i.e., the LTL formula $GF(x = 0)$ holds) in the random walk. As is well known, this property holds almost surely. Given such a verification problem (i.e. a probabilistic program together with an LTL formula), the probabilistic program induces a Markov chain, which typically has an infinite state space. The LTL formula can be translated into acceptance conditions, such as Streett conditions or parity conditions, over the Markov chain. Then, the verification problem asks whether the Markov chain satisfies the associated Streett condition with probability 1 (*qualitative* verification) or with a probability above a given threshold (*quantitative* verification).

Recently, Streett supermartingales [1] are proposed as certificates for such qualitative verification problems. Moreover, when combined with stochastic invariants [6], they can also be used to support quantitative verification [2] [7]. However, the verification power of Streett supermartingales [1] has not been well studied.

Our contributions

We propose several new supermartingale-based certificates for verifying almost sure satisfaction of ω-regular properties: (1) *generalised Streett supermartingales* (GSSMs) and their lexicographic ex-

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ω正規検証のための優マルチンゲールのなす階層構造について

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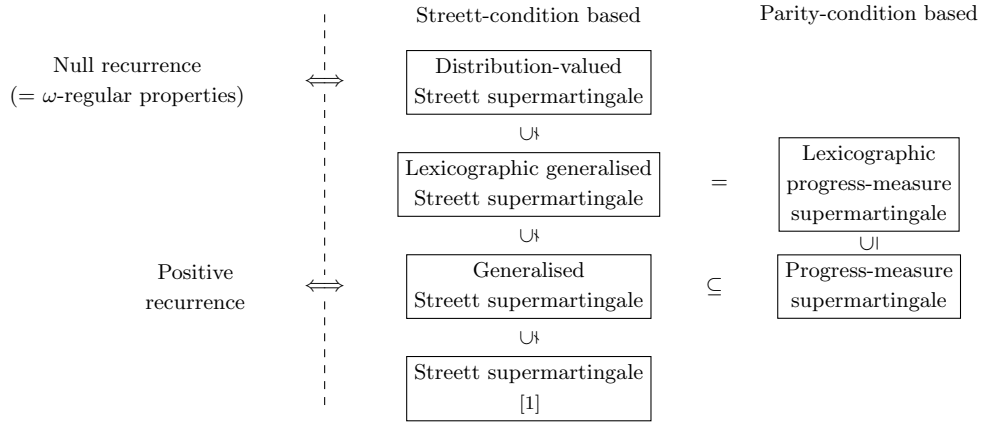

Fig. 1 The hierarchy of supermartingale certificates for almost sure satisfaction of ω-regular properties. The middle column shows supermartingales for Streett conditions, and the right column shows those for parity conditions.

tension (LexGSSMs), (2) *distribution-valued Streett supermartingales* (DVSSMs), and (3) *progress-measure supermartingales* (PMSMs) and their lexicographic extension (LexPMSMs). GSSMs, LexGSSMs, and DVSSMs are derived from least-fixed point characterisations of positive recurrence and null recurrence of Markov chains with Streett conditions; and PMSMs and LexPMSMs are probabilistic extensions of parity progress measures [8].

We systematically investigate the verification power of these certificates and establish a hierarchy among them (Fig. 1). Notably, we show that our certificates are strictly more powerful than Streett supermartingales: we give an ω-regular verification problem that cannot be verified by Streett supermartingales but can be verified by GSSMs.

Example 1. The counterexample is given by the following simple program, which is not even a probabilistic program:

```
while (true) {
    m ≔ n;
    while (m > 0) { m ≔ m − 1 };
    n ≔ n + 1
}
```

Suppose that we want to verify "either the inner loop is executed finitely often, or the outer loop is executed infinitely often", which can naturally be expressed as a Streett acceptance condition. This property clearly holds, and there exists a GSSM that can verify it. However, no Streett supermartingale can verify it.

We also prove completeness of GSSMs for positive recurrence and of DVSSMs for null recurrence, as a consequence of the fixed-point characterisations. The completeness result implies that LexGSSMs, which can reason about some null recurrent Markov chains, are strictly more powerful than GSSMs. Moreover, DVSSMs are, in theory, the most powerful certificates in the sense that for any Markov chain that almost surely satisfies a given ω-regular property, there exists a DVSSM certifying it. As for parity-condition based certificates, we show that LexPMSMs are as powerful as LexGSSMs by constructing mutual translations between LexPMSMs and LexGSSMs.

We propose algorithms for synthesising our new certificates (except for DVSSMs). The algorithm for GSSMs is based on the standard template-based synthesis approach for supermartingales [4], and the algorithms for LexGSSMs and LexPMSMs are based on an extension of the algorithm for lexicographic ranking supermartingales [3]. We have implemented a prototype tool based on these algorithms. Our experiments show that our tool can successfully synthesise certificates for various benchmarks including those that cannot be certified by existing supermartingales.

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