Composing Codensity Bisimulations

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Proving compositionality of behavioral equivalence on state-based systems with respect to algebraic operations is a classical and widely studied problem. We study a categorical formulation of this problem, where operations on state-based systems modeled as coalgebras can be elegantly captured through distributive laws between functors. To prove compositionality, it then suffices to show that this distributive law lifts from sets to relations, giving an explanation of how behavioral equivalence on smaller systems can be combined to obtain behavioral equivalence on the composed system.

In this paper, we refine this approach by focusing on so-called codensity lifting of functors, which gives a very generic presentation of various notions of (bi)similarity as well as quantitative notions such as behavioral metrics on probabilistic systems. The key idea is to use codensity liftings both at the level of algebras and coalgebras, using a new generalization of the codensity lifting. The problem of lifting distributive laws then reduces to the abstract problem of constructing distributive laws between codensity liftings, for which we propose a simplified sufficient condition. Our sufficient condition instantiates to concrete proof methods for compositionality of algebraic operations on various types of state-based systems. We instantiate our results to prove compositionality of qualitative and quantitative properties of deterministic automata. We also explore the limits of our approach by including an example of probabilistic systems, where it is unclear whether the sufficient condition holds, and instead we use our setting to give a direct proof of compositionality.

In addition, we propose a compositional variant of Komorida et al.'s codensity games for bisimilarities. A novel feature of this variant is that it can also compose game invariants, which are subsets of winning positions. Under our sufficient condition of the liftability of distributive laws, composed games give an alternative proof of the preservation of bisimilarities under the composition.

1 Introduction

Bisimilarity and its many variants are fundamental notions of behavioral equivalence on state-based systems. A classical question is whether a given notion of equivalence is *compositional* w.r.t. algebraic operations on these systems, such as parallel composition of concurrent systems, product constructions on automata, or other language constructs. The problem can become particularly challenging when moving from bisimilarity to quantitative notions such as behavioral metrics on probabilistic systems, where it is already non-trivial to even define what the right notion of compositionality is (see, e.g. [20] [10]).

To formulate and address the problem of compositionality of algebraic operations on state-based systems at a high level of generality, a natural formalism is that of *coalgebra* [14], which is parametric in the type of system, as modeled by a "behavior" endofunctor F. A key idea, originating in the seminal work of Turi and Plotkin [25], is that composition operations and in fact whole languages specifying state-based systems arise as *distributive laws* between F and a functor (or monad) T which models the syntax of a bigger programming language. In fact, Turi and Plotkin's *abstract GSOS laws* are

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forms of such distributive laws which guarantee compositionality of strong bisimilarity, generalizing the analogous result for specifications in the *GSOS format* for labeled transition systems [3]. Distributive laws here precisely capture algebra-coalgebra interaction needed for compositionality.

The results of Turi and Plotkin specifically apply to strong bisimilarity, and not directly to other notions such as similarity or behavioral metrics. To prove compositionality in these cases, the key observation is that the task is to define the composition operation at hand not only on state-based systems but also on relations (or, e.g., pseudometrics). This idea has been formalized in the theory of coalgebras by requiring a *lifting* of the distributive law that models the composition operator to a category of relations. The lifting of the behavior functor F then specifies the notion of bisimilarity at hand, and the lifting of the syntax functor T specifies the way that relations on components can be combined into bisimulations on the composite system. It has been shown in [5] at a high level of generality that the existence of such liftings, referred to in this paper as *liftability*, ensures compositionality. The generality there is offered by the use of liftings of the behavior functor in fibrations which goes back to [13], and which allows us to study not only relations but also, for instance, coinductively defined metrics or unary predicates (e.g., [16] [23] [12] [4]). But the main challenge here is to *prove liftability*; this is what we address in the current paper.

We focus on the *codensity lifting* of behavior functors [23], which allows to model a wide variety of coinductive relations and predicates including (bi)similarity but also, for instance, behavioral metrics; these are commonly referred to as *codensity bisimilarity*. In fact, the codensity lifting directly generalizes the celebrated Kantorovich distance between probability distributions used to define, for instance, metrics between probabilistic systems [6]. The codensity lifting is parametric in a collection of modal operators, which makes it on the one hand very flexible and on the other hand much more structured than arbitrary liftings. Moreover, the codensity lifting allows to characterize *codensity games* [16], a generalization of classical bisimilarity games [24] [7] from transition systems to coalgebras, and from strong bisimilarity to a wide variety of coinductively defined relations, metrics and even topologies.

In this paper we study *compositionality of codensity bisimulations* with respect to composition operators on the underlying state-based systems, modeled as coalgebras. We model these composition operators as distributive laws between an n-ary product functor and the behavior functor F, referred to as *one-step composition operators*. We mainly tackle two problems in this paper: 1) How do we lift the n-ary product functor in order to capture non-trivial combination of relations, pseudometrics etc? 2) When is a one-step composition operator liftable?

A key idea for the first problem is to use codensity liftings not only to lift the behavior functor F and get our desired notion of equivalence, but to use another codensity lifting of the product functor, to explain syntactically how relations (or pseudometrics, etc) should be combined from components to the composite system. This combination can be a simple product between relations, but in many cases, such as for behavioral metrics, it needs to be more sophisticated; the flexibility offered by the codensity lifting helps to define the appropriate constructions on relations.

The second problem about liftability then becomes that of proving the existence of a *distributive law between codensity liftings*. To this end, we exhibit a sufficient condition that ensure the existence of this distributive laws, defined in terms of two properties: 1) commutation between the underlying modalities that define codensity liftings, and 2) an *approximation* property, which resembles the one used to prove expressiveness of modal logics in [17]. Underlying our approach is a combination of a new generalization of the codensity lifting beyond endofunctors, to allow to lift product functors, and the adjunction-based decomposition of codensity liftings proposed in [2]. The adjunction-based approach precisely allows us to arrive at our sufficient condition.

We instantiate our approach to pseudometrics on deterministic automata. We also revisit the compositionality of parallel composition w.r.t. behavioral metrics studied in [10]. In this case it is not clear whether our sufficient condition holds; but the framework nevertheless helps to prove liftability and thereby compositionality.

We further study a compositional variant of codensity games. A key observation is that our *composite codensity games* consist of positions that are tuples of positions of codensity games for component systems. This design of the composite game directly leads to the compositionality of game invariants, which characterize winning positions. Assuming our sufficient condition for the liftability of distributive laws, we present an alternative proof of the preservation of bisimilarities along compositions.

In summary, the contributions of this paper are

- a generalization of the codensity lifting beyond endofunctors, which can be used in a special case to lift products in various ways;
- a sufficient condition for proving the existence of a distributive law between codensity liftings, with several detailed examples;
- a composition of codensity games, which also composes game invariants, and an alternative proof of the preservation of codensity bisimilarities under our sufficient condition;

After the preliminaries, we start the paper with a more detailed overview of our approach.

▷ *Related work* There is a wide range of results in the process algebra literature on proving compositionality and on rule formats that guarantee it, usually focused on transition systems (see, e.g., [21] for an overview). A full account is beyond the scope of this paper. We focus instead on generality in the type of models and the type of coinductive predicate. Concerning general coalgebraic frameworks for compositionality, we have already mentioned Turi and Plotkin's abstract GSOS format; the main innovation in the current paper is that we go beyond bisimilarity by employing the codensity lifting.

In [5], it is shown that liftings of distributive laws to fibrations yield so-called *compatibility*, a property that ensures soundness of up-to techniques, and which implies compositionality. On the one hand, if one uses the so-called canonical relation lifting then all distributive laws lift, but this only concerns strong bisimilarity; on the other hand, in [5] examples beyond bisimilarity are studied but liftability there is proven on an ad-hoc basis. In the current paper we identify the codensity lifting as a sweet spot between the (restricted) canonical relation lifting and abstract, unrestricted lifting of functors, and focus instead on conditions that allow to prove liftability.

Many bisimilarity notions are known to be characterized by winning positions of certain safety games, including bisimilarity on Kripke frames [24], probabilistic bisimulation [8] [7], and bisimulation metric [18]. There are several coalgebraic frameworks [18] [9] that captures such a relationship between games and bisimilarities. In this paper, we focus on codensity games [16] that are naturally obtained by codensity liftings and coalgebras.

Codensity liftings are first introduced to lift monads across fibrations [15]. They subsume the Kantorivich metric on the Giry monad. In [22], they are extended to lift endofunctors across $\mathbf{CLat}_{\sqcap}$-fibrations over **Set**, and are shown to be a generalization of Baldan et al.'s Kantorovich lifting [1]. Prior to these papers, the monadic property of the Kantorovich metric is studied in an unpublished manuscript by van Breugel [26]. Codensity liftings for endofunctors can be done in other base categories (such as **Meas** and **Vect**) without modification; see [16]. In [11], the Kantorovich liftings are studied in the context of Lawvere metric space, and are related with other notions of liftings, such as lax extensions and (monotone) predicate liftings.

See [19] for further details.

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