Proof Normalization for Classical Truth-Table Natural Deduction

Kosuke Fukui　Saori Ishii　Koji Nakazawa　Takeshi Tsukada

The truth-table natural deduction was introduced by Geuvers et al. as a framework to construct natural-deduction-style logical systems for general logical connectives accompanied by truth tables that define the semantics of the connectives. They introduced both an intuitionistic variant $\mathrm{IPC}_{\mathcal{C}}$ and a classical variant $\mathrm{CPC}_{\mathcal{C}}$ of the truth-table natural deduction. They also gave a proof normalization for $\mathrm{IPC}_{\mathcal{C}}$, by which some fundamental properties such as consistency and decidability are proved for $\mathrm{IPC}_{\mathcal{C}}$. However, for the classical variant, any proof normalization with nice properties has not been achieved. This paper proposes a new classical variant $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ of the truth-table natural deduction and a proof normalization on $\mathrm{IPC}^{\mu}_{\mathcal{C}}$. $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ is based on Parigot's classical natural deduction, and its proof normalization is obtained by generalizing the lambda-mu calculus. This paper proves that $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ is equivalent to $\mathrm{CPC}_{\mathcal{C}}$ regarding provability and that the proof normalization for $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ satisfies the strong normalization and the subformula property. As corollaries of them, it also gives syntactic proofs of consistency and conservativeness for $\mathrm{IPC}^{\mu}_{\mathcal{C}}$.

1 Introduction

The truth-table natural deduction [1] [2] is a general framework to define natural-deduction-style logical systems for general logical connectives. In it, we can deduce inference rules from truth tables for logical connectives: an elimination rule from each row with the truth value 0 (false), and an introduction rule from each row with the truth value 1 (true). IPC_{C} is an intuitionistic variant of the truth-table natural deduction for the set $\mathcal{C}$ of logical connectives and $\mathrm{CPC}_{\mathcal{C}}$ is its classical variant, which are complete for the intuitionistic Kripke semantics and classical semantics, respectively.

A suitable proof normalization enables us to conclude fundamental properties of logical systems such as consistency, decidability, and conservativeness. Geuvers et al. [2] gave a proof normalization for $\mathrm{IPC}_{\mathcal{C}}$, and van der Giessen [9] [4] showed the subformula property and the strong normalization for it, which imply consistency and decidability of $\mathrm{IPC}_{\mathcal{C}}$. Furthermore, we also obtain conservativeness of $\mathrm{IPC}_{\mathcal{C}}$ by the proof normalization, that is, the extension of $\mathcal{C}$ does not affect the provability for the connectives in $\mathcal{C}$.

On the other hand, as stated in [9] [4], it is still unclear how we can define proof normalization for the classical variant $\mathrm{CPC}_{\mathcal{C}}$ with good properties. They defined $\mathrm{CPC}_{\mathcal{C}}$ in the style of Parigot's free deduction [7] rather than standard natural-deduction style, and it is not easy to define the proof normalization for $\mathrm{CPC}_{\mathcal{C}}$. Geuvers et al. [3] proposed a proof normalization for $\mathrm{CPC}_{\mathcal{C}}$, but its properties such as normalization property have not been studied yet.

In this paper, we propose another classical variant $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ of the truth-table natural deduction, which extends $\mathrm{IPC}_{\mathcal{C}}$ by multiple conclusions in a similar way to Parigot's classical natural deduction introduced [8]. For $\mathrm{IPC}^{\mu}_{\mathcal{C}}$, we can define a proof normalization which is induced from Parigot's lambda-mu calculus [8], which have been widely studied from the view point of both logic and computational calculus. We prove the following properties for $\mathrm{IPC}^{\mu}_{\mathcal{C}}$: (1) Provability of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ is the same as that of $\mathrm{CPC}_{\mathcal{C}}$. (2) Normal proofs in $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ satisfy the subformula property, that is, every formula oc-

---

古典汎用的自然演繹の証明正規化

福井 康介, 石井 沙織, 中澤 巧爾, 名古屋大学, Nagoya University.

塚田 武志, 千葉大学, Chiba University.

curring in a normal proof of a judgment is a subformula of the judgment. (3) The reduction for the proof normalization in $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ is strongly normalizable, that is, there is no infinite reduction sequence. As corollaries for these properties, we have the fundamental properties of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ such as the consistency and conservativeness. These properties can be also proved semantically, and proof normalization gives us syntactic proofs of them.

In order to prove strong normalization, we use the continuation-passing-style (CPS) transformations. In the proof of strong normalization for $\mathrm{IPC}_{\mathcal{C}}$ [9] [4], some continuations are removed during the CPS transformation if the source term contains a particular form of elimination rules, and then we cannot prove the preservation of the strict reduction, that is, the one or more steps reduction. To avoid this problem, van der Giessen splits the cases depending on the form of elimination rules and inserts dummy redexes in the CPS transformation. Moreover their CPS transformation collapses every permutation step, and it is necessary to prove the strong normalization of permutation conversion separately.

However, such a naive case splitting does not work for $\mathrm{IPC}^{\mu}_{\mathcal{C}}$, and we adopt the continuation-and-garbage-passing-style (CGPS) transformation, which was introduced by Ikeda et al. [5] to solve the erasing continuation problem uniformly for several calculi. In the CGPS transformation, every continuation which occurs during the CPS transformation is accumulated in the garbage parts, which play the role of the dummy redexes. Moreover, permutation steps and additional μ-reduction steps are preserved as garbage-disposal steps, and hence it is not necessary to prove strong normalization of these reductions separately. The technique with CGPS transformation also gives a simpler strong normalization proof for $\mathrm{IPC}_{\mathcal{C}}$ because we need not prove the strong normalization separately.

2 Truth-table natural deduction and proof normalization

2.1 $\mathrm{IPC}_{\mathcal{C}}$ and $\mathrm{CPC}_{\mathcal{C}}$

In this section, we recall two variants of the truth-table natural deduction. One is the intuitionistic variant $\mathrm{IPC}_{\mathcal{C}}$, and the other is the classical variant $\mathrm{CPC}_{\mathcal{C}}$ [1]. Here, $\mathcal{C}$ is a set of logical connectives, each of which has a fixed arity, and are supposed to be accompanied with a truth table t_c. We use 0 (false) and 1 (true) for the truth values, and $\mathbb{B}$ for the set $\{0, 1\}$. The truth table t_c for an n-ary connective is given as a mapping from $\mathbb{B}^n$ to $\mathbb{B}$.

Definition 2.1 (Formulas and judgments)**.** The *formulas* on $\mathcal{C}$ are defined as

$$A ::= p \mid c(A_1, \cdots, A_n),$$

where p is a propositional variable, and c is a connective in $\mathcal{C}$ whose arity is n. For a set Γ of formulas and a formula A, we call $\Gamma \vdash A$ a *judgment*. For the set $\Gamma = \{A_1, \dots, A_m\}$, we write $A_1, \dots, A_m \vdash A$ for $\Gamma \vdash A$. We write Γ_1, Γ_2 for the union of Γ_1 and Γ_2. In particular, when Γ_2 is a singleton $\{A\}$, we write Γ_1, A for the union. When Γ is empty, we write $\vdash A$ for $\Gamma \vdash A$.

Definition 2.2 (Inference rules)**.** The inference rules of the truth-table natural deduction for an n-ary connective c are in Fig. 1, where $\vec{b}$ is a sequence $b_1, \dots, b_n$ of truth values $b_i \in \mathbb{B}$, and $\vec{A} = A_1, \dots, A_n$ is a sequence of formulas.

Definition 2.3 (*$IPC_{\mathcal{C}}$* and *$CPC_{\mathcal{C}}$* [1])**.** 1. $\mathrm{IPC}_{\mathcal{C}}$ consists of ax and the inference rules $\mathsf{el}^c(\vec{b})$ for $c \in \mathcal{C}$ and $\vec{b}$ such that $t_c(\vec{b}) = 0$, and $\mathsf{in}^c(\vec{b})$ for $c \in \mathcal{C}$ and $\vec{b}$ such that $t_c(\vec{b}) = 1$.

2. $\mathrm{CPC}_{\mathcal{C}}$ consists of ax and the inference rules $\mathsf{el}^c(\vec{b})$ for $c \in \mathcal{C}$ and $\vec{b}$ such that $t_c(\vec{b}) = 0$, and $\mathsf{inC}^c(\vec{b})$ for $c \in \mathcal{C}$ and $\vec{b}$ such that $t_c(\vec{b}) = 1$.

Example 2.4. *Consider* xor *with the truth table defined as* $t_{\mathsf{xor}}(1, 1) = t_{\mathsf{xor}}(0, 0) = 0$ *and* $t_{\mathsf{xor}}(1, 0) = t_{\mathsf{xor}}(0, 1) = 1$*. The inference rules of* $\mathrm{IPC}_{\{\mathsf{xor}\}}$ *are* ax *and the the rules in Fig. 2, and* $\mathrm{CPC}_{\{\mathsf{xor}\}}$ *is obtained by replacing the two introduction rules by the rules in Fig. 3.*

For general $\mathcal{C}$, these natural deductions satisfy the soundness and the completeness.

Theorem 2.5 ([1] [9])**.** *1.* $\mathrm{IPC}_{\mathcal{C}}$ *is sound and complete with respect to the intuitionistic Kripke semantics, where the truth in each possible world is defined by the given truth tables.*

2. $\mathrm{CPC}_{\mathcal{C}}$ *is sound and complete with respect to the classical (two-valued) semantics, where the truth is defined by the given truth tables.*

2.2 Proof normalization for $\mathrm{IPC}_{\mathcal{C}}$

Next, we recall the proof normalization for $\mathrm{IPC}_{\mathcal{C}}$ [2]. Following [2], we give the reduction system on proof terms based on the Curry-Howard correspondence, but we adopt a slightly different notation

1. Axiom:

$$\frac{}{\Gamma, A \vdash A}\ \mathsf{ax}$$

2. Elimination rule: (D is an arbitrary formula.)

$$\frac{\Gamma \vdash c(\vec{A}) \quad \cdots \quad \overset{(\text{for } b_i = 1)}{\Gamma \vdash A_i} \quad \cdots \quad \overset{(\text{for } b_j = 0)}{\Gamma, A_j \vdash D} \quad \cdots}{\Gamma \vdash D}\ \mathsf{el}^c(\vec{b})$$

3. Intuitionistic introduction rule:

$$\frac{\cdots \quad \overset{(\text{for } b_i = 1)}{\Gamma \vdash A_i} \quad \cdots \quad \overset{(\text{for } b_j = 0)}{\Gamma, A_j \vdash c(\vec{A})} \quad \cdots}{\Gamma \vdash c(\vec{A})}\ \mathsf{in}^c(\vec{b})$$

4. Classical introduction rule: (D is an arbitrary formula.)

$$\frac{\Gamma, c(\vec{A}) \vdash D \quad \cdots \quad \overset{(\text{for } b_i = 1)}{\Gamma \vdash A_i} \quad \cdots \quad \overset{(\text{for } b_j = 0)}{\Gamma, A_j \vdash D} \quad \cdots}{\Gamma \vdash D}\ \mathsf{inC}^c(\vec{b})$$

Fig. 1 Inference rules of truth-table natural deductions

$$\frac{\Gamma \vdash A_1 \mathsf{xor} A_2 \quad \Gamma \vdash A_1 \quad \Gamma \vdash A_2}{\Gamma \vdash D}\ \mathsf{el}^{\mathsf{xor}}(1,1) \qquad \frac{\Gamma \vdash A_1 \quad \Gamma, A_2 \vdash A_1 \mathsf{xor} A_2}{\Gamma \vdash A_1 \mathsf{xor} A_2}\ \mathsf{in}^{\mathsf{xor}}(1,0)$$

$$\frac{\Gamma, A_1 \vdash A_1 \mathsf{xor} A_2 \quad \Gamma \vdash A_2}{\Gamma \vdash A_1 \mathsf{xor} A_2}\ \mathsf{in}^{\mathsf{xor}}(0,1) \qquad \frac{\Gamma \vdash A_1 \mathsf{xor} A_2 \quad \Gamma, A_1 \vdash D \quad \Gamma, A_2 \vdash D}{\Gamma \vdash D}\ \mathsf{el}^{\mathsf{xor}}(0,0)$$

Fig. 2 Inference rules of $\mathrm{IPC}_{\{\mathsf{xor}\}}$

$$\frac{\Gamma, A_1 \mathsf{xor} A_2 \vdash D \quad \Gamma \vdash A_1 \quad \Gamma, A_2 \vdash D}{\Gamma \vdash D}\ \mathsf{inC}^{\mathsf{xor}}(1,0) \qquad \frac{\Gamma, A_1 \mathsf{xor} A_2 \vdash D \quad \Gamma, A_1 \vdash D \quad \Gamma \vdash A_2}{\Gamma \vdash D}\ \mathsf{inC}^{\mathsf{xor}}(0,1)$$

Fig. 3 Introduction rules of $\mathrm{CPC}_{\{\mathsf{xor}\}}$

from [2], where they assign proof terms after sorting the assumptions of each inference rules by the value of b_i, whereas we fix the order of the assumptions.

Definition 2.6 (Proof terms)**.** The *proof terms* for $\mathrm{IPC}_{\mathcal{C}}$ are defined as

$$M, N ::= x \mid M \cdot_{\mathsf{el}^c(\vec{b})} [\vec{P}] \mid \{\vec{P}\}_{\mathsf{in}^c(\vec{b})}$$
$$P, Q ::= M \mid (x)M,$$

where x is a term variable and $\vec{P}$ is a sequence $P_1, \ldots, P_n$. We often omit the annotation by names of inference rules when they are not important or obvious from contexts. We call expressions of the form $(x)M$ *abstracted terms*, and the occurrences of x in M are supposed to be bound. As usual, we identify terms which differ only in names of bound variables. We write $M \cdot [\vec{P}] \cdot [\vec{Q}]$ for $(M \cdot [\vec{P}]) \cdot [\vec{Q}]$.

The term assignment rules for the proofs of $\mathrm{IPC}_{\mathcal{C}}$ are in Fig. 4, where the judgment is extended to the form $\Gamma \vdash M : A$ and Γ is extended to the set of pairs $x : A$ of term variables and formulas. We assume that Γ always satisfies the condition: if $x : A, x : B \in \Gamma$, then $A = B$.

Definition 2.7 (Reduction rules for $\mathrm{IPC}_{\mathcal{C}}$ [2])**.** The reduction rules on the proof terms of $\mathrm{IPC}_{\mathcal{C}}$ are in Fig. 5. $M[x := N]$ on the right-hand side of each β-rule is the ordinary capture-avoiding substitution. $P_i @ [\vec{Q}]$ on the right-hand side of the π-rule is defined as N if $P_i = N$ and $(x)(N \cdot [\vec{Q}])$ if $P_i = (x)N$.

The relation $M \to N$ means that N is obtained by replacing a subterm of M by one of the reduction rules. A subterm of M which matches the left-hand side of one of the reduction rules is called a redex. $\to^+$ and $\to^*$ are the transitive closure and the reflexive transitive closure of $\to$, respectively. When we use only the β-rules, we write $\to_\beta$, $\to_\beta^+$, and $\to_\beta^*$. Similarly, we use $\to_\pi$, and so on. M is said *$\beta\pi$-normal* when there is no N such that $M \to N$.

Note that, a proof term of the form $\{\vec{P}\}_{\mathsf{in}^c(\vec{b})} \cdot_{\mathsf{el}^c(\vec{b'})} [\vec{Q}]$ is always a β-redex. In fact, since $t_c(\vec{b}) = 1$ and $t_c(\vec{b'}) = 0$, we have $b_i \neq b'_i$ for some i, and we can apply one of the β-rules at the index i.

For the proof normalization of $\mathrm{IPC}_{\mathcal{C}}$, the following have been proved [9] [4]. In this existing work, subject reduction is not explicitly shown, but it is proved in a straightforward way.

Theorem 2.8 (Proof normalization of $\mathrm{IPC}_{\mathcal{C}}$)**.** *In*

$$\frac{}{\Gamma, x : A \vdash x : A}\ \mathsf{ax}$$

$$\frac{\Gamma \vdash M : c(\vec{A}) \quad \cdots \quad \overset{(\text{for } b_i = 1)}{\Gamma \vdash N_i : A_i} \quad \cdots \quad \overset{(\text{for } b_j = 0)}{\Gamma, x_j : A_j \vdash N_j : D} \quad \cdots}{\Gamma \vdash M \cdot_{\mathsf{el}^c(\vec{b})} [\cdots N_i \cdots (x_j)N_j \cdots] : D}\ \mathsf{el}^c(\vec{b})$$

$$\frac{\cdots \quad \overset{(\text{for } b_i = 1)}{\Gamma \vdash N_i : A_i} \quad \cdots \quad \overset{(\text{for } b_j = 0)}{\Gamma, x_j : A_j \vdash N_j : c(\vec{A})} \quad \cdots}{\Gamma \vdash \{\cdots N_i \cdots (x_j)N_j \cdots\}_{\mathsf{in}^c(\vec{b})} : c(\vec{A})}\ \mathsf{in}^c(\vec{b})$$

Fig. 4 Term assignment rules of $\mathrm{IPC}_{\mathcal{C}}$

$$\{\cdots M_i \cdots\}_{\mathsf{in}^c(\vec{b})} \cdot_{\mathsf{el}^c(\vec{b'})} [\cdots (x_i)N_i \cdots] \mapsto_\beta N_i[x_i := M_i] \qquad (\text{for } b_i = 1 \text{ and } b'_i = 0)$$

$$\{\cdots (x_i)M_i \cdots\}_{\mathsf{in}^c(\vec{b})} \cdot_{\mathsf{el}^c(\vec{b'})} [\cdots N_i \cdots] \mapsto_\beta M_i[x_i := N_i] \cdot_{\mathsf{el}^c(\vec{b'})} [\cdots N_i \cdots] \qquad (\text{for } b_i = 0 \text{ and } b'_i = 1)$$

$$(M \cdot [P_1, \cdots, P_n]) \cdot [\vec{Q}] \mapsto_\pi M \cdot [P_1 @ [\vec{Q}], \cdots, P_n @ [\vec{Q}]]$$

Fig. 5 Reduction rules of $\mathrm{IPC}_{\mathcal{C}}$

$\mathrm{IPC}_{\mathcal{C}}$*, the following hold.*

1. (Subject reduction) If $\Gamma \vdash M : A$ *is provable and* $M \to N$ *holds, then* $\Gamma \vdash N : A$ *is provable.*

2. (Subformula property [9]) Every formula which occurs in a $\beta\pi$*-normal proof of* $\Gamma \vdash M : A$ *is a subformula of either* Γ *or* A*.*

3. (Strong normalization [9][4]) If $\Gamma \vdash M : A$ *is provable, there is no infinite* $\beta\pi$*-reduction sequence from* M*.*

We will discuss van der Giessen's proof of strong normalization in Section 4.

As corollaries of these properties, we can prove consistency and conservativeness of $\mathrm{IPC}_{\mathcal{C}}$.

Corollary 2.9. *1.* $\mathrm{IPC}_{\mathcal{C}}$ *is consistent, that is, there is a judgment which is not provable in* $\mathrm{IPC}_{\mathcal{C}}$*.*

2. Suppose $\mathcal{C} \subseteq \mathcal{C}'$ *and that* Γ *and* A *contain only formulas on* $\mathcal{C}$*. Then,* $\Gamma \vdash A$ *is provable in* $\mathrm{IPC}_{\mathcal{C}}$ *if and only if it is provable in* $\mathrm{IPC}_{\mathcal{C}'}$*.*

3 $\mathrm{IPC}^{\mu}_{\mathcal{C}}$: another classical truth-table natural deduction

In this section, we give another variant $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ of classical truth-table natural deduction based on the classical natural deduction in [8]. We show that the provability in $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ the same as that in $\mathrm{CPC}_{\mathcal{C}}$ and we can define a proof normalization for $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ as an extension of the lambda-mu calculus [8], which has been well studied.

3.1 Definition of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$

Definition 3.1 ($\mathrm{IPC}^{\mu}_{\mathcal{C}}$)**.** The judgments of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ are the expressions of the form $\Gamma \vdash A; \Delta$, where Γ and Δ are sets of formulas. The inference rules of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ are those of $\mathrm{IPC}_{\mathcal{C}}$ extended to the judgments of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ in a straightforward way, and the following additional rules.

$$\frac{\Gamma \vdash A; B, \Delta}{\Gamma \vdash B; A, \Delta}\ \mu_1 \qquad \frac{\Gamma \vdash A; A, \Delta}{\Gamma \vdash A; \Delta}\ \mu_2$$

Theorem 3.2 (Soundness of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$)**.** *If* $\Gamma \vdash A; \Delta$ *is provable in* $\mathrm{IPC}^{\mu}_{\mathcal{C}}$*, then for any valuation of the classical (two-valued) semantics such that every formula in* Γ *is true, there is a formula in* $\{A\} \cup \Delta$ *which is true in the valuation.*

Proof. It is proved by induction on proofs. □

Theorem 3.3 (Equivalence to $\mathrm{CPC}_{\mathcal{C}}$)**.** $\Gamma \vdash A$ *is provable in* $\mathrm{CPC}_{\mathcal{C}}$ *if and only if* $\Gamma \vdash A;$ *is provable in* $\mathrm{IPC}^{\mu}_{\mathcal{C}}$*.*

Proof. (From $\mathrm{CPC}_{\mathcal{C}}$ to $\mathrm{IPC}^{\mu}_{\mathcal{C}}$) We can show that the classical introduction rules are admissible in $\mathrm{IPC}^{\mu}_{\mathcal{C}}$. We use the fact that the cut rule and the weakening rules are admissible in $\mathrm{IPC}^{\mu}_{\mathcal{C}}$.

(From $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ to $\mathrm{CPC}_{\mathcal{C}}$) If $\Gamma \vdash A;$ is provable in $\mathrm{IPC}^{\mu}_{\mathcal{C}}$, it is valid in the classical semantics by the soundness of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$. By the completeness of $\mathrm{CPC}_{\mathcal{C}}$, $\Gamma \vdash A$ is provable in $\mathrm{CPC}_{\mathcal{C}}$. □

Corollary 3.4 (Completeness of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$)**.** $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ *is sound and complete with respect to the classical (two-valued) semantics, where the truth is defined by the given truth tables.*

3.2 Proof normalization of $\mathrm{IPC}^\mu_\mathcal{C}$

Definition 3.5 (Proof terms of $\mathrm{IPC}^\mu_\mathcal{C}$)**.** The proof terms of $\mathrm{IPC}^\mu_\mathcal{C}$ are defined as follows.

$$M ::= x \mid M \cdot_{\mathsf{el}^c(\vec{b})} [\vec{P}] \mid \{\vec{P}\}_{\mathsf{in}^c(\vec{b})} \mid \mu\alpha.\beta M$$
$$P ::= M \mid (x)M,$$

where α and β are μ-variables, which are different sorts of variables from the term variables. Occurrences of α in $\mu\alpha.\beta M$ are supposed to be bound.

The term assignment rules of $\mathrm{IPC}^\mu_\mathcal{C}$ are extended from those of $\mathrm{IPC}_\mathcal{C}$ as follows. The judgments are extended to the form $\Gamma \vdash A; \Delta$, where Γ is a set of pairs $x : A$ and Δ is a set of pairs $\alpha : A$. The rules for the axiom, the elimination rules, and the intuitionistic introduction rules are analogous to $\mathrm{IPC}_\mathcal{C}$. The term assignment for μ_1 and μ_2 are the following.

$$\frac{\Gamma \vdash M : A;\ \beta : B, \Delta}{\Gamma \vdash \mu\beta.\alpha M : B;\ \alpha : A, \Delta}\ \mu_1$$
$$\frac{\Gamma \vdash M : A;\ \alpha : A, \Delta}{\Gamma \vdash \mu\alpha.\alpha M : A;\ \Delta}\ \mu_2$$

Definition 3.6 (Structural substitution)**.** $M[\alpha X := \alpha X \cdot [\vec{P}]]$ is defined as the proof term obtained by recursively replacing each occurrence αN by $\alpha(N \cdot [\vec{P}])$.

Definition 3.7 (Reduction rules for $\mathrm{IPC}^\mu_\mathcal{C}$)**.** The reduction of $\mathrm{IPC}^\mu_\mathcal{C}$ is defined by the β- and π-rules of $\mathrm{IPC}_\mathcal{C}$, and the following μ-rule.

$$(\mu\alpha.\beta M) \cdot [\vec{P}] \mapsto_\mu \mu\alpha.(\beta M)[\alpha X := \alpha X \cdot [\vec{P}]]$$

In $\mathrm{IPC}^\mu_\mathcal{C}$, $M \to N$ means that N is obtained by replacing a subterm of M by one of the β-, π-, and μ-rules.

We can show the following properties on the proof normalization of $\mathrm{IPC}^\mu_\mathcal{C}$.

Theorem 3.8 (Proof normalization of $\mathrm{IPC}^\mu_\mathcal{C}$)**.** *In $\mathrm{IPC}^\mu_\mathcal{C}$, the following hold.*

1. (Subject reduction) If $\Gamma \vdash M : A; \Delta$ is provable and $M \to N$ holds, then $\Gamma \vdash N : A; \Delta$ is provable.

2. (Subformula property) Every formula which occurs in a $\beta\pi\mu$-normal proof of $\Gamma \vdash M : A; \Delta$ is a subformula of either Γ, A or Δ.

3. (Strong normalization) If $\Gamma \vdash M : A; \Delta$ is provable, there is no infinite $\beta\pi\mu$-reduction sequence from M.

Proof. 1. It is proved in a standard way.

2. It is proved by induction on the normal proofs. By the definition of the reduction rules, for every elimination rule $M \cdot [\vec{P}]$ in a $\beta\pi\mu$-normal proof, M must be a term variable.

3. This will be proved in Section 5. □

Similarly to the case of $\mathrm{IPC}_\mathcal{C}$, we have the following corollaries.

Corollary 3.9. *1. $\mathrm{IPC}^\mu_\mathcal{C}$ is consistent, that is, there is a judgment which is not provable in $\mathrm{IPC}^\mu_\mathcal{C}$.*

2. Suppose $\mathcal{C} \subseteq \mathcal{C}'$ and that Γ, A, and Δ contain only formulas on $\mathcal{C}$. Then, $\Gamma \vdash A; \Delta$ is provable in $\mathrm{IPC}^\mu_\mathcal{C}$ if and only if it is provable in $\mathrm{IPC}^\mu_{\mathcal{C}'}$.

Furthermore, these hold also for $\mathrm{CPC}_\mathcal{C}$, which are proved by the corollaries together with Theorem 3.3.

4 Strong normalization of $\mathrm{IPC}_\mathcal{C}$

In this section, we recall the proof in [9] [4] of the strong normalization of $\mathrm{IPC}_\mathcal{C}$.

4.1 Simply typed parallel λ-calculus

First, we recall the extended typed λ-calculus $\mathrm{p}\lambda^\to$ [9], which is the target calculus of the CPS transformations.

Definition 4.1 ($\mathrm{p}\lambda^\to$)**.** The types and the terms of $\mathrm{p}\lambda^\to$ are defined as follows.

$$A ::= a \mid A \to A$$
$$M ::= x \mid (MM) \mid \lambda x.M \mid (M_1 \mid\mid \cdots \mid\mid M_n),$$

where a is an atomic type and $n \geq 2$. The terms of the form $(M_1 \mid\mid \cdots \mid\mid M_n)$ are called *parallel terms*. The typing rules of $\mathrm{p}\lambda^\to$ are the following.

$$\frac{}{\Gamma, x : A \vdash x : A} \qquad \frac{\Gamma, x : A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B}$$
$$\frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash MN : B}$$
$$\frac{\Gamma \vdash M_1 : A \quad \cdots \quad \Gamma \vdash M_n : A}{\Gamma \vdash (M_1 \mid\mid \cdots \mid\mid M_n) : A}$$

Definition 4.2. The reduction rules of $\mathrm{p}\lambda^\to$ are the following

$$(\lambda x.M)N \mapsto M[x := N]$$
$$(M_1 \mid\mid \cdots \mid\mid M_n)N \mapsto (M_1 N \mid\mid \cdots \mid\mid M_n N)$$

$M \to N$ means that N is obtained by replacing a subterm of M by one of the reduction rules. $\to^+$ and $\to^*$ are the transitive closure and the reflexive transitive closure, respectively.

The following relation $M \sqsubseteq N$ intuitively means that N is obtained by adding some terms in parallel terms in M.

Definition 4.3. The relation $\sqsubseteq$ on the terms of $\mathrm{p}\lambda^\to$ is inductively defined as follows.

- $M \sqsubseteq M$.
- If $N \sqsubseteq M_i$ for some i, then $N \sqsubseteq (M_1 \mid\mid \cdots \mid\mid M_n)$.
- If $N_i \sqsubseteq M_i$ for all i, then $(N_1 \mid\mid \cdots \mid\mid N_n) \sqsubseteq (M_1 \mid\mid \cdots \mid\mid M_n)$.
- If $N \sqsubseteq M$, then $\lambda x.N \sqsubseteq \lambda x.M$.

- If $N_1 \sqsubseteq M_1$ and $N_2 \sqsubseteq M_2$, then $N_1 N_2 \sqsubseteq M_1 M_2$.

The following lemma will play an important role in the proofs of the strong normalization.

Lemma 4.4. *If $M \sqsubseteq M'$ and $M \to N$, then there exists N' such that $M' \to^+ N'$ and $N \sqsubseteq N'$.*

The strong normalization of $\mathrm{p}\lambda^{\to}$ has been proved in [9].

Theorem 4.5 (Strong normalization of $\mathrm{p}\lambda^{\to}$ [9] [4])**.** *There is no infinite reduction sequence from a typable term in* $\mathrm{p}\lambda^{\to}$*.*

4.2 CPS transformation of $\mathrm{IPC}_{\mathcal{C}}$

The CPS transformation defined later preserves the typability of terms through the following negative transformation.

Definition 4.6 (Negative transformation)**.** Let o be a fixed atomic type in $\mathrm{p}\lambda^{\to}$. We use the abbreviation $\neg_o(A_1, \cdots, A_n) = A_1 \to \cdots \to A_n \to o$. In particular, $\neg_o A$ denotes $A \to o$. For $b \in \mathbb{B}$, we define $\xi_b A$ by $\xi_1 A = A$ and $\xi_0 A = \neg_o A$. The *negative translation* $\overline{A}$ from the formulas of $\mathrm{IPC}_{\mathcal{C}}$ to the types of $\mathrm{p}\lambda^{\to}$ is as follows.

- $\overline{p} = \neg_o \neg_o p$
- $\overline{c(A_1, \cdots, A_n)} = \neg_o \neg_o (E_{\vec{b}^1}, \cdots, E_{\vec{b}^k})$, where $(\vec{b}^1, \cdots, \vec{b}^k)$ is the list of all sequences $\vec{b}$ such that $t_c(\vec{b}) = 0$, and for each $\vec{b}^i = (b^i_1, \cdots, b^i_n)$, $E_{\vec{b}^i}$ is defined as $\neg_o(\xi_{b^i_1} \overline{A_1}, \cdots, \xi_{b^i_n} \overline{A_n})$.

A naive definition of the CPS transformation corresponding to the negative transformation is defined as $\overline{M} = \lambda k.(M : k)$, where the transformation $M : K$ is defined in Fig. 6.

This CPS transformation preserves typability, but does not preserve strict reduction, since there are two cases in which some redexes are erased during the transformation: (1) When $\vec{P}$ in $M \cdot [\vec{P}] : K$ contains no abstracted term, K is erased. (2) For $\{\vec{P}\}_{\mathsf{in}^c(\vec{b})} : K$, when there is no elimination rule $\mathsf{el}^c(\vec{b}')$ such that $b_i \neq b'_i$, P_i is erased. Hence, the redexes in the erased parts are also erased during the transformation. The first case is discussed in [6] [5] as the erasing-continuation problem.

To solve this problem, in [9] [4] they introduced a case distinction depending on whether the erasing-continuation problem arises or not, and inserted some dummy redexes.

Definition 4.7 (CPS transformation for $\mathrm{IPC}_{\mathcal{C}}$)**.** The CPS transformation for $\mathrm{IPC}_{\mathcal{C}}$ is defined as $\overline{M} = \lambda k.(M : k)$, where the transformation $M : K$ is defined as follows.

For this definition, they can show that if $M \to_\beta N$ and $\overline{M} \sqsubseteq M'$ hold, then there is N' such that $\overline{N} \sqsubseteq N'$ and $M' \to^+ N'$, and hence we can reduce the strong normalization of β-reduction to that of $\mathrm{p}\lambda^{\to}$.

They need a further analysis for the π-reduction. The π-reductions are divided into two cases: for a π-redex $M \cdot [\vec{P}] \cdot [\vec{Q}]$, it is called positive if $\vec{P}$ contains an abstracted term, and negative otherwise. Then, we can show (1) if $M \to_\pi N$ is positive, then we have $\overline{M} = \overline{N}$, (2) negative π-reduction steps can be postponed, and (3) The π-reduction is strongly normalizing. By these facts, they proved the strong normalization of the $\beta\pi$-reduction in $\mathrm{IPC}_{\mathcal{C}}$ [9] [4].

5 Strong normalization of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$

The strong normalization of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ can be proved by a CPS transformation in a similar way to the case of $\mathrm{IPC}_{\mathcal{C}}$. However, the naive case distinction like Definition 4.7 does not work for $\mathrm{IPC}^{\mu}_{\mathcal{C}}$. Following the standard CPS transformation for the $\lambda\mu$-calculus, we extend the definition of $M : K$ as $\mu\alpha.M : K = (M : \lambda x.x)[k_\alpha := K]$ and $\alpha M : K = M : k_\alpha$, where we suppose a fixed variable k_α in $\mathrm{p}\lambda^{\to}$ for each μ-variable α. In this transformation, the erasing-continuation problem arises in $\mu\alpha.M : K$ if M contains no free occurrence of α, and hence the case distinction should be done depending on whether M contains free α or not in $\mu\alpha.M : K$. However, the reduction $\mu\alpha.M \to \mu\alpha.M'$ may erase α in M, and then $\mu\alpha.M : K$ and $\mu\alpha.M' : K$ are defined in the different cases. In fact, the transformation defined by such a naive case distinction does not preserve the reduction.

5.1 CGPS transformation

In our proof, we adopt the *continuation-and-garbage-passing style* (CGPS) transformation [5], in which all continuations (the terms denoted by K) are accumulated in the "garbage part" to keep all of redexes in source terms. Furthermore, the garbage part also plays a role measuring some depth of nested elimination rules, and the π- and the μ-reduction steps are also preserved strictly as garbage-disposal steps.

Definition 5.1 (CGPS transformation for $\mathrm{IPC}^{\mu}_{\mathcal{C}}$)**.**

$$x : K = xK$$
$$M \cdot_{\mathsf{el}^c(\vec{b}^l)} [\cdots N_i \cdots (x_j)N_j \cdots] : K = M : \lambda g_1 \cdots g_k.g_l \cdots \overline{N_i} \cdots (\lambda x_j.(N_j : K)) \cdots$$
$$\{\cdots N_i \cdots (x_j)N_j \cdots\}_{\mathsf{in}^c(\vec{b})} : K = KL^K_{\vec{b},\vec{b}^1} \cdots L^K_{\vec{b},\vec{b}^k},$$

where $(\vec{b}^1, \cdots, \vec{b}^k)$ is the list of all sequences $\vec{b}$ such that $t_c(\vec{b}) = 0$, and for each $\vec{b}^l$, $L^K_{\vec{b},\vec{b}^l}$ is the parallel term which consists of the following terms.

$$\lambda h_1 \cdots h_n.h_i\overline{N_i} \qquad \text{(for } b_i = 1 \text{ and } b^l_i = 0\text{)}$$
$$\lambda h_1 \cdots h_n.(\lambda x_j.(N_j : K))h_j \qquad \text{(for } b_j = 0 \text{ and } b^l_j = 1\text{)}$$

Fig. 6 A naive definition of CPS transformation

$$x : K = xK$$
$$M \cdot_{\mathsf{el}^c(\vec{b}^l)} [\cdots N_i \cdots (x_j)N_j \cdots] : K = M : \lambda g_1 \cdots g_k.g_l \cdots \overline{N_i} \cdots (\lambda x_j.(N_j : K)) \cdots$$
$$M \cdot_{\mathsf{el}^c(\vec{b}^l)} [\cdots N_i \cdots] : K = M : (\lambda h g_1 \cdots g_k.g_l \cdots \overline{N_i} \cdots)K$$
$$\text{([}\cdots N_i \cdots\text{] contains no abstracted term)}$$
$$\{\cdots N_i \cdots (x_j)N_j \cdots\}_{\mathsf{in}^c(\vec{b})} : K = ((\lambda q_1 \cdots q_n.K) \cdots \overline{N_i} \cdots \lambda x_j(N_j : K) \cdots)L^K_{\vec{b},\vec{b}^1} \cdots L^K_{\vec{b},\vec{b}^k},$$

where $L^K_{\vec{b},\vec{b}^i}$ are defined as in Fig. 6.

Fig. 7 CPS transformation for $\mathrm{IPC}_{\mathcal{C}}$

For terms M and N of $\mathrm{p}\lambda^{\to}$, we write $\langle\!\langle M; N \rangle\!\rangle$ to denote $(\lambda x.M)N$, where x is a variable which does not occur in M. We also use the notation $\langle\!\langle M_1; M_2; \cdots; M_n \rangle\!\rangle = \langle\!\langle \cdots \langle\!\langle M_1; M_2 \rangle\!\rangle \cdots; M_n \rangle\!\rangle$. The CGPS transformation for $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ is defined as $\overline{\overline{M}} = \lambda gk.(M : g, k)$, where the transformation $M : G, K$ is defined as Fig. 8

5.2 Preservation of typability

The CGPS transformation preserves typability of terms through the negative transformation extended with the type $\top$ of garbage.

Definition 5.2 (Negative transformation)**.** Let $\top$ be the type $o \to o$. The *negative transformation* is defined as $\overline{\overline{A}} = \top \to \neg_o A^{\bullet}$ for an $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ formula A, where $A^{\bullet}$ is defined as $p^{\bullet} = \neg_o p$ and $c(A_1, \cdots, A_n)^{\bullet} = \neg_o(E_{\vec{b}^1}, \cdots, E_{\vec{b}^k})$, where $(\vec{b}^1, \cdots, \vec{b}^k)$ is the list of all sequences $\vec{b}$ such that $t_c(\vec{b}) = 0$, and for each $\vec{b}^i = (b^i_1, \cdots, b^i_n)$, we define $E_{\vec{b}^i} = \neg_o(\xi_{b^i_1}\overline{\overline{A_1}}, \cdots, \xi_{b^i_n}\overline{\overline{A_n}})$.

Proposition 5.3 (Preservation of typability)**.** *If* $\Gamma \vdash M : A; \Delta$ *is provable in* $\mathrm{IPC}^{\mu}_{\mathcal{C}}$*, then* $\overline{\overline{\Gamma}}, \Delta^{\bullet}, \Delta^{\top} \vdash \overline{\overline{M}} : \overline{\overline{A}}$ *is provable in* $\mathrm{p}\lambda^{\to}$*.*

Proof. It is proved by induction on the proofs in $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ simultaneously with the following statement: For any terms K and G of $\mathrm{p}\lambda^{\to}$, if $\Gamma \vdash M : A; \Delta$, $\overline{\overline{\Gamma}}, \Delta^{\bullet}, \Delta^{\top} \vdash G : \top$, and $\overline{\overline{\Gamma}}, \Delta^{\bullet}, \Delta^{\top} \vdash K : A^{\bullet}$, then we have $\overline{\overline{\Gamma}}, \Delta^{\bullet}, \Delta^{\top} \vdash (M : G, K) : o$. ☐

5.3 Preservation of strict reduction

The last part of the strong normalization proof is to prove that the CGPS transformation preserves the strict reduction. To prove this, we need the following lemmas.

Lemma 5.4. *1. If $G \to G'$ holds, then we have $M : G, K \to^{+} M : G', K$.*

2. If $K \to K'$ holds, then we have $M : G, K \to^{*} M : G, K'$.

3. If $G \sqsubseteq G'$ and $K \sqsubseteq K'$ hold, then we have $M : G, K \sqsubseteq M : G', K'$.

Proof. These are proved by induction on M. ☐

Proposition 5.5. *If $M \to_{\beta\pi\mu} M'$ and $\overline{\overline{M}} \sqsubseteq N$ hold, then there exists N' such that $\overline{\overline{M'}} \sqsubseteq N'$ and $N \to^{+} N'$.*

By this proposition, the strong normalization of $\mathrm{IPC}^{\mu}_{\mathcal{C}}$ is reduced to that of $\mathrm{p}\lambda^{\to}$ as Fig. 9

References

[1] Geuvers, H. and Hurkens, T.: Deriving natural deduction rules from truth tables, *Logic and Its Applications—7th Indian Conference (ICLA 2017)*, LNCS, Vol. 10119, 2017, pp. 123–138.

[2] Geuvers, H. and Hurkens, T.: Proof terms for generalized natural deduction, *23rd International Conference on Types for Proofs and Programs (TYPES 2017)*, LIPIcs, Vol. 104, 2018, pp. 3:1–3:39.

[3] Geuvers, H. and Hurkens, T.: Proof terms for

$$
\begin{aligned}
x : G, K &:= xGK, \\
M \cdot_{\mathsf{el}^c(\vec{b}^{\,l})} [\vec{P}] : G, K &= M : \langle\!\langle G; \mathcal{K}_l^{G,K}(\vec{P}) \rangle\!\rangle, \mathcal{K}_l^{G,K}(\vec{P}), \\
\{\cdots N_i \cdots (x_j)N_j \cdots\}_{\mathsf{in}^c(\vec{b})} : G, K &= \langle\!\langle K L^{G,K}_{\vec{b},\vec{b}^{1}} \cdots L^{G,K}_{\vec{b},\vec{b}^{k}}; \cdots ; \overline{\overline{N_i}}; \cdots ; \lambda x_j.(N_j : G, K); \cdots ; G \rangle\!\rangle, \\
\mu\alpha.M : G, K &= (M : g_\alpha, \lambda x.x)[g_\alpha := G, k_\alpha := K], \\
\alpha M : G, K &= \langle\!\langle (M : g_\alpha, k_\alpha); G \rangle\!\rangle,
\end{aligned}
$$

where $(\vec{b}^{1}, \cdots, \vec{b}^{k})$ is the list of all sequences $\vec{b}$ such that $t_c(\vec{b}) = 0$, and

$$\mathcal{K}_l^{G,K}(\cdots N_i \cdots (x_j)N_j \cdots) = \lambda g_1 \cdots g_k.g_l \cdots \overline{\overline{N_i}} \cdots (\lambda x_j.(N_j : G, K)) \cdots ,$$

and for each l, $L^{G,K}_{\vec{b},\vec{b}^{l}}$ is the parallel term which consists of the following terms.

$$
\begin{array}{ll}
\lambda h_1 \cdots h_n.h_i \overline{\overline{N_i}} & \text{(for } b_i = 1 \text{ and } b_i^l = 0\text{)} \\
\lambda h_1 \cdots h_n.(\lambda x_j.(N_j : G, K))h_j & \text{(for } b_j = 0 \text{ and } b_j^l = 1\text{)}
\end{array}
$$

Fig. 8 CGPS transformation for $\mathrm{IPC}^{\mu}_{\mathcal{C}}$

$$
\begin{array}{ccccccccccc}
M_1 & \to & M_2 & \to & M_3 & \to & M_4 & \to & \cdots & (\mathrm{IPC}^{\mu}_{\mathcal{C}}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & & \\
\overline{\overline{M_1}} & & \overline{\overline{M_2}} & & \overline{\overline{M_3}} & & \overline{\overline{M_4}} & & \cdots & (\mathrm{p}\lambda^{\to}) \\
\sqsupseteq & & \sqsupseteq & & \sqsupseteq & & \sqsupseteq & & & \\
N_1 & \to^{+} & N_2 & \to^{+} & N_3 & \to^{+} & N_4 & \to^{+} & \cdots & (\mathrm{p}\lambda^{\to})
\end{array}
$$

Fig. 9 Proof of strong normalization

generalized classical natural deduction, *26th International Conference on Types for Proofs and Programs (TYPES 2020)*, 2020, pp. 156–158.

[4] Geuvers, H., van der Giessen, I., and Hurkens, T.: Strong normalization for truth table natural deduction, *Fundamenta Informaticae*, Vol. 170, No. 1-3(2019), pp. 139–176.

[5] Ikeda, S. and Nakazawa, K.: Strong normalization proofs by CPS-translations, *Information Processing Letters*, Vol. 99(2006), pp. 163–170.

[6] Nakazawa, K. and Tatsuta, M.: Strong normalization proof with CPS-translation for second order classical natural deduction, *The Journal of Symbolic Logic*, Vol. 68, No. 3(2003), pp. 851–859. Corrigendum is available in: The Journal of Symbolic Logic 68 (4) (2003) 1415–1416.

[7] Parigot, M.: Free deduction: an analysis of "computations" in classical logic, *2nd Russian Conference on Logic Programming (RCLP 1991)*, LNCS, Vol. 592, 1992, pp. 361–380.

[8] Parigot, M.: $\lambda\mu$-calculus: An algorithmic interpretation of classical natural deduction, *Logic Programming and Automated Reasoning,International Conference LPAR'92*, LNCS, Vol. 624, 1992, pp. 190–201.

[9] van der Giessen, I.: Natural deduction derived from truth tables, Master's thesis, Radboud Universiteit Nijmegen, 2018.