Fibered Fibration Categories

Taichi Uemura

We show that, for a fibered category whose base category is a type theoretic fibration category, fiberwise and total structures of type theoretic fibration categories coincide.

1 Introduction

Homotopy type theory [12] is a recent subject that combines Martin-Löf's dependent type theory with homotopy theory. It provides an abstract language for proving homotopy-theoretic theorems. Among abstract languages of this sort including Quillen model categories [7] [3] and various other models for $(\infty, 1)$-categories [6], homotopy type theory has unique tools that are convenient for some purposes.

One such tool is **higher inductive types**. They give a simple way to construct spaces such as spheres, tori and other cell complexes. We can define maps from a higher inductive type *by recursion* and prove theorems on a higher inductive type *by induction*, just like functions and theorems on natural numbers.

Another tool is Voevodsky's **univalence axiom**. This states that equivalent types are identical. With this axiom we can prove that isomorphic groups are equal, equivalent categories are equal, isometric Hilbert spaces are equal and any other "isomorphisms" with certain structure can be replaced by equalities.

Combining higher inductive types and the univalence axiom, we can construct *fibrations* over higher inductive types and prove more theorems. For example, it can be shown that the fundamental group of $\mathbb{S}^1$ is $\mathbb{Z}$, using the "universal cover" on $\mathbb{S}^1$.

Homotopy type theory provides new technical tools for algebraic topologists, but what can we say about type theory itself? To analyze type theory, we want a correct notion of models of homotopy type theory. Some concrete and abstract models already exist. Hofmann and Streicher constructed the groupoid model of a restricted type theory, in which every type is 1-truncated, and proved that the groupoid of all small sets is a univalent universe in this model [2]. Kapulkin, Lumsdaine and Voevodsky proved that the category of simplicial sets models the univalence axiom, under the assumption that there is an inaccessible cardinal [5]. Shulman introduced *type theoretic fibration categories* as categorical semantics of Martin-Löf's type theory and gave a construction of type theoretic fibration categories that preserves univalence [9].

In this paper, we examine Shulman's construction from a different point of view. Shulman showed that if $\mathbb{C}$ is a type theoretic fibration category then so is a full subcategory of its arrow category $\mathbb{C}^2$ and generalized this result to $\mathbb{C}^I$ for any inverse category I. However, there is another generalization, regarding $\mathbb{C}^2$ as a *fibered category* $\mathrm{cod} : \mathbb{C}^2 \to \mathbb{C}$ such that every fiber has type theoretic structures. Fibered categories have used as models of *logical predicates* [4] [1]. Thus fibered categories between type theoretic fibration categories will give logical predicates for homotopy type theory.

In fibered category theory, it often happens that if each fiber has structures such as finite limits, Cartesian closed structure and monoidal structure,

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上村 太一, 京都大学数理解析研究所, Research Institute for Mathematical Science, Kyoto University.

then so does the total category and sometimes the converse holds. In this paper, we consider categorical structures that correspond to identity types and dependent products, and show that such structures on the total category and each fiber coincide.

Similar approaches have been taken by Roig [8] and Stanculescu [11] in the study of model categories. They considered weak factorization systems and model structures on bifibered categories and gave a construction from fiberwise structures to total structures.

Organization. We begin section 2 by defining the basic categorical structures which corresponds to the type theoretic operations: dependent products and identity types. We call a category that has identity types a *fibration category.* Fibration categories that has also dependent products are exactly Shulman's type theoretic fibration categories.

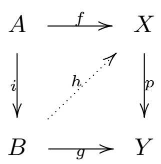
In section 3, we introduce two notions of fibered categories between (type theoretic) fibration categories. One has type theoretic structures on *each fiber* and the other has them on *total category.* Then we show the main result: fiberwise and total type theoretic structures are equivalent.

2 Type Theoretic Fibration Categories

In this section we define fibration categories and type theoretic fibration categories, and state some basic properties of them.

2.1 Lifting properties

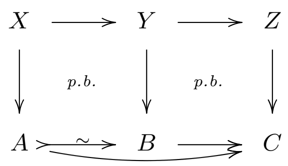
Definition 2.1. *Let $i : A \to B$ and $p : X \to Y$ in a category. i has the* left lifting property *with respect to p (p has the* right lifting property *with respect to i) if for all $f : A \to X$ and $g : B \to Y$ such that $p \circ f = g \circ i$, there exists an $h : B \to X$ such that $h \circ i = f$ and $p \circ h = g$.*



2.2 Fibration categories

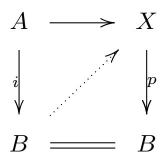
Definition 2.2. *A* fibration category *is a category $\mathbb{C}$ equipped with a terminal object 1 and a subcategory $\mathcal{F} \subset \mathbb{C}$ satisfying the conditions below. Here a morphism in $\mathcal{F}$ is called a* fibration *and denoted by a two headed arrow $X \twoheadrightarrow Y$, and a morphism that has the left lifting property with respect to all fibrations is called an* acyclic cofibration *and denoted by $X \overset{\sim}{\rightarrowtail} Y$.*

1. *All isomorphisms and all morphisms with codomain 1 are fibrations.*
2. *Fibrations are stable under pullbacks: that means, if $p : X \to J$ is a fibration and $f : I \to J$ is any morphism, then there exists a pullback f^*X of X along f and $f^*X \to I$ is again a fibration.*
3. *Every morphism factors as an acyclic cofibration followed by a fibration.*
4. *In the following diagram,*



if $B \to C$ and $A \to C$ are fibrations, $A \to B$ is an acyclic cofibration, and both squares are pullbacks, then $X \to Y$ is an acyclic cofibration.

Remark 2.3. To show that a morphism $i : A \to B$ in a fibration category $\mathbb{C}$ is an acyclic cofibration, we have only to show that, for every fibration $p : X \twoheadrightarrow B$, there exists a lift of the form



since fibrations are stable under pullbacks.

Example 2.4.
1. If a category $\mathbb{C}$ has finite limits, then $\mathbb{C}$ is a fibration category with fibrations all morphisms. In this case, acyclic cofibrations are exactly isomorphisms.
2. Let $\mathbb{C}$ be a model category whose cofibrations are exactly monomorphisms. Then the full subcategory of all fibrant objects of $\mathbb{C}$ is a fibration category.
3. Let $\mathbb{C}$ be a fibration category. Write $(\mathbb{C}/A)_{\mathrm{f}}$ for the full subcategory of $\mathbb{C}/A$ whose objects are fibrations over A. $(\mathbb{C}/A)_{\mathrm{f}}$ is a fibration category: $f : X \to Y$ is a fibration if it is a fibration in $\mathbb{C}$.

Lemma 2.5. *If $i : A \overset{\sim}{\rightarrowtail} B$ is an acyclic cofibration, then there exists a morphism $p : B \to A$ such* *that* $pi = 1$.

Proof. Use lifting property for the fibration $A \twoheadrightarrow 1$. $\square$

Lemma 2.6. *If* gf *and* g *are acyclic cofibrations, then so is* f.

Proof. See Lemma 2.4 of [10]. $\square$

2. 3 Type theoretic fibration categories

Definition 2.7. *A* type theoretic fibration category *is a fibration category* $\mathbb{C}$ *satisfying the following condition.*

1. *For any fibrations* $g : A \twoheadrightarrow B$ *and* $f : X \twoheadrightarrow A$, *there exists a fibration* $\Pi_g f : \Pi_g X \twoheadrightarrow B$ *such that, for every* $h : Z \to B$,

$$\mathbb{C}/B\,(Z, \Pi_g X) \cong \mathbb{C}/A\,(Z \times_B A, X) \qquad (1)$$

As usual, $\Pi_g f$ is, if exists, unique up to unique isomorphism and, if we choose $\Pi_g f$ for each f, Π_g determines a functor from $(\mathbb{C}/A)_{\mathrm{f}}$ to $(\mathbb{C}/B)_{\mathrm{f}}$.

Remark 2.8. Under the condition 1 of definition 2.7, 4 of definition 2.2 follows from other axioms of fibration category (Lemma 2.3 of [10]).

Lemma 2.9. *In a type theoretic fibration category* $\mathbb{C}$, *pullback along a fibration preserves acyclic cofibrations.*

Proof. Apply isomorphism 1 for an acyclic cofibration $h : Z \overset{\sim}{\rightarrowtail} B$. $\square$

Lemma 2.10. *If* $g : A \twoheadrightarrow B$ *is a fibration in a type theoretic fibration category* $\mathbb{C}$, *then* $\Pi_g : (\mathbb{C}/A)_f \to (\mathbb{C}/B)_f$ *preserves fibrations.*

Proof. See Lemma 3.1 of [10]. $\square$

2. 4 Strong fibration functors

Definition 2.11. *A functor between fibration categories is a* strong fibration functor *if it preserves*

- *Terminal objects*
- *Fibrations*
- *Pullbacks of fibrations and*
- *Acyclic cofibrations*

Example 2.12. Re-indexing functor $u^* : (\mathbb{C}/B)_{\mathrm{f}} \to (\mathbb{C}/A)_{\mathrm{f}}$ is a strong fibration functor.

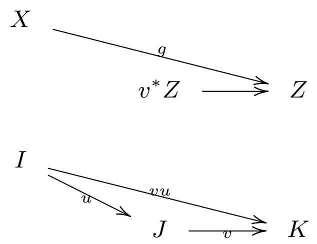
3 Fibered Fibration Categories

In this section, we define total and fiberwise structures of type theory on a fibered category, and show that they are equivalent. Fibered categories with total fibration structures are called fibered fibration categories. For fiberwise type theoretic structures, we introduce a temporary term "fiberwise fibration categories". However, these two terms are very confusing and thus we will officially use "fibered fibration categories" after proving that these notions coincide.

3. 1 Total structures

Definition 3.1. *A* fibered fibration category *is a strong fibration functor* $p : \mathbb{E} \to \mathbb{B}$ *between fibration categories such that*

1. p *is a fibered category*
2. *Every Cartesian morphism over a fibration is a fibration.*
3. *In the following diagram in* $\mathbb{E} \to \mathbb{B}$,

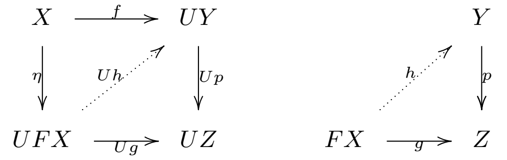


   if g, v *and* u *are fibrations, then the induced morphism* $X \to v^*Z$ *over* u *is a fibration.*
4. *Every Cartesian morphism over an acyclic cofibration is an acyclic cofibration.*

3. 2 Fiberwise structures

Definition 3.2. *Let* $U : \mathbb{D} \to \mathbb{C}$ *be a strong fibration functor between fibration categories.* U has a homotopy left adjoint *if, for every object* X *in* $\mathbb{C}$, *there exist an object* FX *in* $\mathbb{D}$ *and* $\eta : X \to UFX$ *such that, for every* $g : FX \to Z$, *every fibration* $p : Y \to Z$ *and every morphism* $f : X \to UY$ *such that* $Up \circ f = Ug \circ \eta$, *there exists a morphism*

$h : FX \to Y$ *such that* $p \circ h = g$ *and* $Uh \circ \eta = f$.



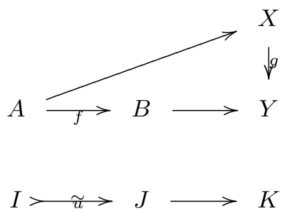
Remark 3.3. Taking $Z = 1$ in the definition 3.2, we have an "almost" right adjoint F to U except that morphisms $h : FX \to Y$ such that $Uh \circ \eta = f$ are not unique. However, we can say that they are *homotopic*.

Definition 3.4. *A* fiberwise fibration category *is a fibered category* $p : \mathbb{E} \to \mathbb{B}$ *such that*

1. $\mathbb{B}$ *and all fiber* $\mathbb{E}_I$ *are fibration categories.*
2. *For every* $u : I \to J$ *in* $\mathbb{B}$, *the re-indexing functor* $u^* : \mathbb{E}_J \to \mathbb{E}_I$ *is a strong fibration functor.*
3. *For every acyclic cofibration* $u : I \overset{\sim}{\rightarrowtail} J$ *in* $\mathbb{B}$, *the re-indexing functor* u^* *has a homotopy left adjoint.*

Remark 3.5. Since re-indexing functors preserve fibrations, 3 of definition 3.4 is equivalent to the following.

- For every acyclic cofibration $u : I \overset{\sim}{\rightarrowtail} J$ and every Cartesian morphism $f : A \to B$ over u, if $g : X \twoheadrightarrow Y$ is a fibration in the following diagram in $\mathbb{E} \to \mathbb{B}$,

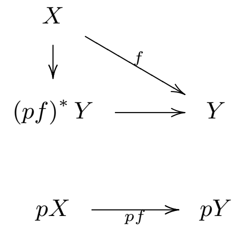


then there exists a filling morphism $h : B \to X$.

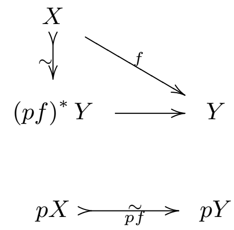
3.3 Reedy fibrations

In [9], Shulman defined *Reedy fibrations* in arrow category $\mathbb{C}^2$ of a type theoretic fibration category $\mathbb{C}$. We generalize it for a fiberwise fibration category in a natural way.

Definition 3.6. *Suppose* $p : \mathbb{E} \to \mathbb{B}$ *is a fiberwise fibration category. A morphism* $f : X \to Y$ *in* $\mathbb{E}$ *is a* Reedy fibration *if* pf *is a fibration in* $\mathbb{B}$ *and the induced morphism* $X \to (pf)^* Y$ *is a fibration in* $\mathbb{E}_{pX}$.



f *is a* Reedy acyclic cofibration *if* pf *is an acyclic cofibration in* $\mathbb{B}$ *and the induced morphism* $X \to (pf)^* Y$ *is an acyclic cofibration in* $\mathbb{E}_{pX}$.



3.4 From fiberwise to total structures

Theorem 3.7. *Suppose* $p : \mathbb{E} \to \mathbb{B}$ *is a fiberwise fibration category. Then* $\mathbb{E}$ *has a structure of fibration category whose fibrations are exactly Reedy fibrations and* p *is a fibered fibration category.*

To show this theorem, we need several lemmas.

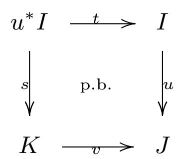
Lemma 3.8. *Let* $1_{\mathbb{B}}$ *be a terminal object in* $\mathbb{B}$. *Then a terminal object in* $\mathbb{E}_{1_{\mathbb{B}}}$ *is a terminal object in* $\mathbb{E}$.

Lemma 3.9. *Suppose* $u : I \overset{\sim}{\rightarrowtail} J$ *is an acyclic cofibration and* A *is an object over* I. *Then there exists a Cartesian morphism* $\eta : A \to u_! A$ *over* u.

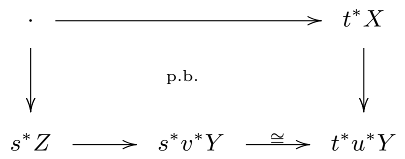
Proof. By Lemma 2.5, there exists a morphism $v : J \to I$ such that $vu = 1$. Thus $A \cong u^* v^* A$ over I, and so we have a Cartesian morphism $A \to v^* A$ over u. $\square$

Lemma 3.10. *Reedy fibrations are stable under pullbacks.*

Proof. Suppose $p : X \to Y$ is a Reedy fibration over a fibration $u : I \twoheadrightarrow J$ and $f : Z \to Y$ is a morphism over $v : K \to J$. First, we have a pullback in $\mathbb{B}$:



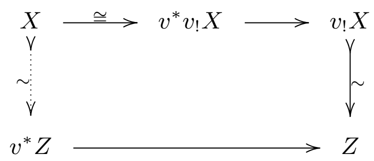
Then we can take a pullback in $\mathbb{E}_{u^*I}$:



This gives a desired pullback. □

Lemma 3.11. *Every morphism f in $\mathbb{E}$ factors as a Reedy acyclic cofibration followed by a Reedy fibration.*

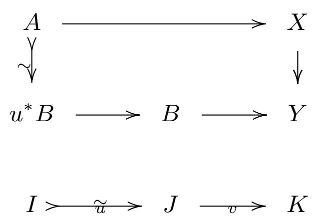
Proof. Let $pf = u : I \to J$. Since $\mathbb{B}$ is a fibration category, $u = w \circ v$ for some acyclic cofibration $v : I \overset{\sim}{\rightarrowtail} K$ and some fibration $w : K \twoheadrightarrow J$. Then we have a morphism $v_!X \to w^*Y$ in $\mathbb{E}_K$. Taking a factorization in $\mathbb{E}_K$, we have an acyclic cofibration $v_!X \overset{\sim}{\rightarrowtail} Z$ followed by a fibration $Z \twoheadrightarrow w^*Y$. Since $X \to v_!X$ is a Cartesian morphism over v and v^* preserves acyclic cofibrations, we have a Reedy acyclic cofibration as the following diagram.



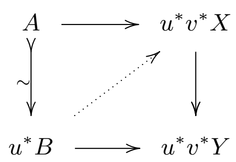
□

Lemma 3.12. *A Reedy acyclic cofibration satisfies the lifting property of 3.5.*

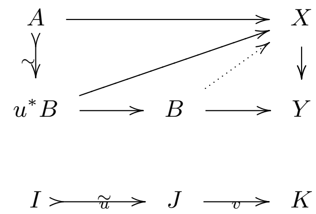
Proof. Suppose $i : A \to B$ is a Reedy acyclic cofibration over an acyclic cofibration $u : I \overset{\sim}{\rightarrowtail} J$. Given a diagram,



we have first a filling morphism in $\mathbb{E}_I$



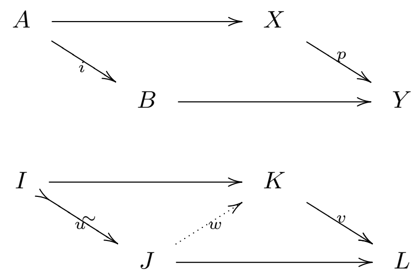
and then there exists a desired filling morphism,



by 3 of definition 3.4. □

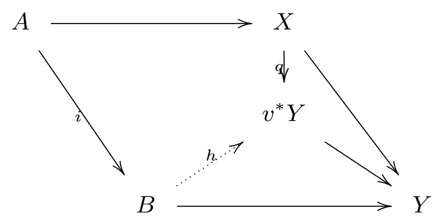
Lemma 3.13. *A Reedy acyclic cofibration has the left lifting property with respect to all Reedy fibrations.*

Proof. Given a diagram in $\mathbb{E} \to \mathbb{B}$

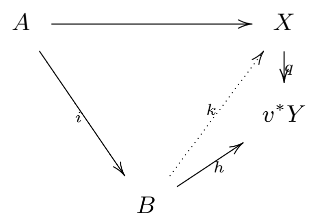


where i is a Reedy acyclic cofibration and p is a Reedy fibration. By definition 3.6, v is a fibration and the induced morphism $q : X \to v^*Y$ is a fibration in $\mathbb{E}_K$. By 3.12, u is an acyclic cofibrations, and i satisfies the lifting property of 3.5.

First, we have a filling morphism $w : J \to K$ in $\mathbb{B}$. Then, by Cartesianness, we have the following morphism h over w:



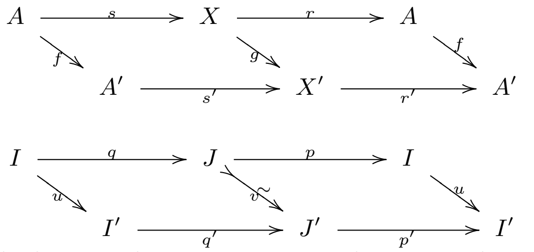
Since $i : A \to B$ satisfies the lifting property of 3.5, we have the following morphism k over w:



This gives a desired filling morphism. □

Lemma 3.14. *Reedy acyclic cofibrations are close* *under retracts.*

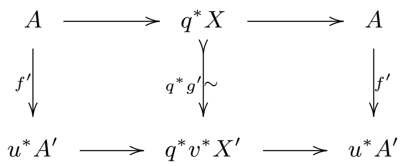
Proof. Suppose in the following diagram in $\mathbb{E} \to \mathbb{B}$,



the horizontal compositions are identities and g is a Reedy acyclic cofibration over an acyclic cofibration v. Write $f' : A \to u^*A'$ and $g' : X \to v^*X'$ for the induced morphisms.

First, since acyclic cofibrations are closed under retracts in a fibration category, u is an acyclic cofibration.

By re-indexing, we have the following diagram in $\mathbb{E}_I$.

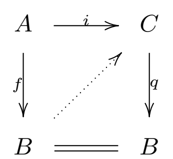


Since re-indexing functors preserve acyclic cofibrations, $q^*g' : q^*X \to q^*v^*X'$ is an acyclic cofibration, and so is its retract f'. □

Lemma 3.15. *In* $\mathbb{E}$*, a morphism has the left lifting property with respect to all Reedy fibrations if and only if it is a Reedy acyclic cofibration.*

Proof. The "if" part is shown in 3.13. We show the "only if" part.

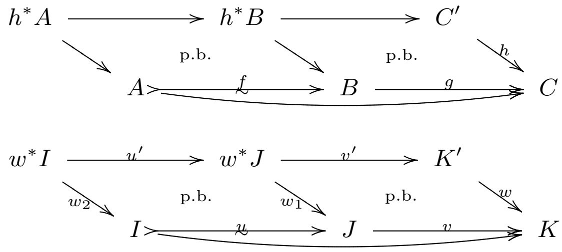
Suppose $f : A \to B$ has the left lifting property with respect to all Reedy fibrations. By 3.11, f factors as a Reedy acyclic cofibration $i : A \to C$ followed by a Reedy fibration $q : C \to B$. Then there exists a filling morphism



Thus f is a retract of i and is a Reedy acyclic cofibration by 3.14. □

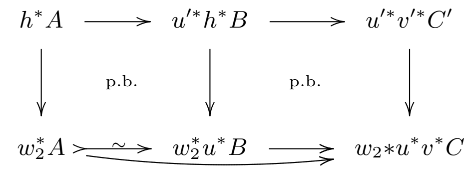
Lemma 3.16. $\mathbb{E}$ *satisfies 4 of definition 2.2.*

Proof. Suppose in the following pullback diagram in $\mathbb{E} \to \mathbb{B}$,



f is a Reedy acyclic cofibration and g and gf are a Reedy fibrations. First, $w^*I \to w^*J$ is an acyclic cofibration.

By re-indexing, we have the following pullback diagram in $\mathbb{E}_{w^*I}$.



Thus the induced morphism $h^*A \to u'^*h^*B$ is an acyclic cofibration. □

3.4.1 Proof of the theorem 3.7

The previous lemmas show that $\mathbb{E}$ is a fibration category whose fibrations are exactly Reedy fibrations. The other conditions of 3.1 is clear by definitions.

3.5 From total to fiberwise structures

Theorem 3.17. *Suppose* $p : \mathbb{E} \to \mathbb{B}$ *is a fibered fibration category. Then* p *is a fiberwise fibration category whose fibers* $\mathbb{E}_I$ *have the following structure.*

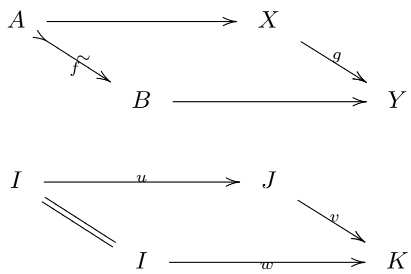
1. *A morphism in* $\mathbb{E}_I$ *is a fibration if and only if it is a fibration in* $\mathbb{E}$.
2. $\pi_I^* 1_{\mathbb{E}}$ *is a terminal object in* $\mathbb{E}_I$*, where* $\pi_I : I \to 1_{\mathbb{B}}$ *is the unique morphism to a terminal object in* $\mathbb{B}$.

To prove this theorem, we need some lemmas.

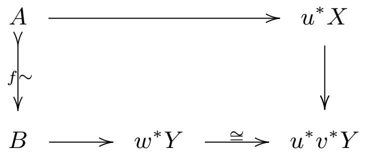
Lemma 3.18. *A morphism in* $\mathbb{E}_I$ *is an acyclic cofibration in* $\mathbb{E}_I$ *if and only if it is an acyclic cofibration in* $\mathbb{E}$.

Proof. Let $f : A \to B$ over I be an acyclic cofibration in $\mathbb{E}_I$. Suppose in the following diagram in

$\mathbb{E} \to \mathbb{B}$,

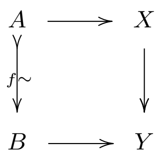


g is a fibration. By re-indexing, we have a diagram in $\mathbb{E}_I$.



Thus we have a filling morphism $B \to u^*X$.

Conversely, suppose $f : A \to B$ over I is an acyclic cofibration in $\mathbb{E}$. Given a diagram in $\mathbb{E}_I$,



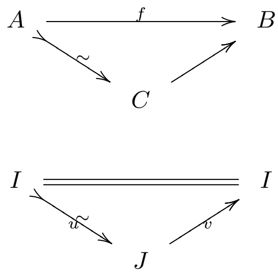
we have a filling morphism $h : B \to X$ in $\mathbb{E}$. Since all other morphisms are mapped to the identity by $p : \mathbb{E} \to \mathbb{B}$, p maps h to the identity. Thus we have a filling morphism in $\mathbb{E}_I$. □

Lemma 3.19. *$f : A \to B$ over an acyclic cofibration $u : I \overset{\sim}{\rightarrowtail} J$ is an acyclic cofibration if and only if the induced morphism $A \to u^*B$ is an acyclic cofibration.*

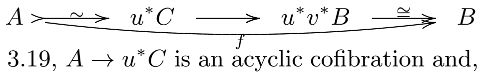
Proof. By 4 of 3.1, the Cartesian lifting $u^*B \to B$ is an acyclic cofibration. Thus, by Lemma 2.6, f is an acyclic cofibration if and only if the induced morphism $A \to u^*B$ is an acyclic cofibration. □

Lemma 3.20. *In each fiber $\mathbb{E}_I$, every map factors as an acyclic cofibration followed by a fibration.*

Proof. Given a morphism $f : A \to B$ over I, we have the following factorization in $\mathbb{E}$.



By re-indexing, we have the following factorization in $\mathbb{E}_I$.



By 3.19, $A \to u^*C$ is an acyclic cofibration and, by 3 of Definition 3.1, $C \to v^*B$ is a fibration. □

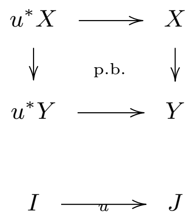
Lemma 3.21. *Each fiber $\mathbb{E}_I$ is a fibration category.*

Proof. By the previous lemmas, we have a factorization in $\mathbb{E}_I$. All the other conditions hold since a pullback in $\mathbb{E}_I$ is taken in $\mathbb{E}$. □

Lemma 3.22. *For every $u : I \to J$, the re-indexing functor $u^* : \mathbb{E}_J \to \mathbb{E}_I$ is a strong fibration functor.*

Proof. Clearly u^* preserves terminal objects.

In the following diagram in $\mathbb{E} \to \mathbb{B}$,



the above square is a pullback. Since a morphism over I is a fibration or an acyclic cofibration in $\mathbb{E}_I$ if and only if it is a fibration or an acyclic cofibration respectively in $\mathbb{E}$, u^* preserves fibrations, pullbacks of fibrations and acyclic cofibrations between fibrations. □

3.5.1 Proof of the theorem 3.17

Combine the previous lemmas. Note that 3 of 3.4 follows from 4 of 3.1.

3.6 Equivalence of fiberwise and total structures

By definition, it is clear that the constructions in theorem 3.7 and 3.17 are mutually inverse.

3.7 Total and fiberwise dependent products

Finally, we consider dependent products on a fibered fibration category. The next theorem is a generalization of both Shulman's construction of dependent products on $\mathbb{C}^2$ (found in the proof of Theorem 8.8 of [9]) and CCC-structure on the total category (Proposition 9.2.4 of [4]).

Theorem 3.23. *Suppose $p : \mathbb{E} \to \mathbb{B}$ is a fibered fibration category and $\mathbb{B}$ is a type theoretic fibration category. Then the followings are equivalent.*

1. *$\mathbb{E}$ is a type theoretic fibration category and p preserves dependent products.*
2. *Each fiber $\mathbb{E}_I$ is a type theoretic fibration category, re-indexing functors preserve dependent products and, for any fibration $u : I \twoheadrightarrow J$ in $\mathbb{B}$, the re-indexing functor $u^* : \mathbb{E}_J \to \mathbb{E}_I$ has a right adjoint u_* preserving fibrations, and satisfies the Beck-Chevalley condition.*

Proof. (1 to 2) Each fiber $\mathbb{E}_I$ has dependent products, by taking products in the total category $\mathbb{E}$.

For a fibration $u : I \twoheadrightarrow J$ in $\mathbb{B}$, denote $\hat{u} : 1_I \twoheadrightarrow 1_J$ for the Cartesian morphism over u between fibered terminal objects. Then the assignment $\mathbb{E}_I \ni X \mapsto \Pi_{\hat{u}} X \in \mathbb{E}_J$ is a right adjoint to u^*.

(2 to 1) Suppose $f : A \twoheadrightarrow X$ and $g : X \twoheadrightarrow Y$ is fibrations over $u : K \twoheadrightarrow I$ and $v : I \twoheadrightarrow J$ respectively. We construct a fibration $\Pi_g f : \Pi_g A \twoheadrightarrow Y$ over $\Pi_v u : \Pi_v K \twoheadrightarrow J$.

Write $f' : A \twoheadrightarrow u^* X$ and $g' : X \twoheadrightarrow v^* Y$ for the induced morphisms. Consider the diagram

$$\begin{array}{llll}
\varepsilon^* A & & & \\
\downarrow{\scriptstyle \varepsilon^* f'} & & & \\
\varepsilon^* u^* X & & \varepsilon^* \Pi_{u^* g'} A & \\
 & \searrow{\scriptstyle \varepsilon^* u^* g'} & \downarrow & \\
 & & \varepsilon^* u^* v^* Y & \xrightarrow{\tilde{v}} (\Pi_v u)^* Y \\
 & & & \\
 & & v^* \Pi_v K & \xrightarrow{\hat{v}} \Pi_v K
\end{array}$$

where $\varepsilon : v^* \Pi_v K \to K$ is the counit of the adjunction $v^* \dashv \Pi_v$. We have a Cartesian morphism $\tilde{v}$ over $\hat{v}$ since $vu\varepsilon = \hat{v} \Pi_v u$. Then pulling $\hat{v}_* \varepsilon^* \Pi_{u^* g'} A$ back along the unit $\eta : (\Pi_v u)^* Y \to \hat{v}_* \varepsilon^* u^* v^* Y$ of the adjunction $\hat{v}^* \dashv \hat{v}_*$, we have a fibration $\eta^* \hat{v}_* \varepsilon^* \Pi_{u^* g'} A \twoheadrightarrow (\Pi_v u)^* Y$ in $\mathbb{E}_{\Pi_v K}$.

$$\begin{array}{ccc}
\eta^* \hat{v}_* \varepsilon^* \Pi_{u^* g'} A & \longrightarrow & \hat{v}_* \varepsilon^* \Pi_{u^* g'} A \\
\downarrow & \text{p.b.} & \downarrow \\
(\Pi_v u)^* Y & \xrightarrow{\eta} & \hat{v}_* \varepsilon^* u^* v^* Y
\end{array}$$

This gives a desired dependent product. □

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References

[1] C. Hermida. *Fibrations, Logical Predicates and Indeterminates.* PhD thesis, University of Edinburgh, 1993.

[2] Martin Hofmann and Thomas Streicher. The groupoid interpretation of type theory. In *Twenty-five years of constructive type theory (Venice, 1995)*, volume 36 of *Oxford Logic Guides*, pages 83–111. Oxford Univ. Press, New York, 1998.

[3] M. Hovey. *Model Categories.* Mathematical surveys and monographs. American Mathematical Society, 2007.

[4] Bart Jacobs. *Categorical logic and type theory.* Elsevier Science, 1st edition, December 1999.

[5] Chris Kapulkin, Peter L. Lumsdaine, and Vladimir Voevodsky. The Simplicial Model of Univalent Foundations, April 2014. arXiv:1211.2851.

[6] Jacob Lurie. *Higher topos theory.* Princeton University Press, 2009.

[7] Daniel G. Quillen. *Homotopical algebra.* Springer, 1967.

[8] Agustí Roig. Model category structures in bifibred categories. *Journal of Pure and Applied Algebra*, 95(2):203–223, August 1994.

[9] Michael Shulman. Univalence for inverse diagrams and homotopy canonicity. *Mathematical Structures in Computer Science*, 25(05):1203–1277, June 2015.

[10] Michael Shulman. Univalence for inverse EI diagrams, August 2015. arXive: 1508.02410.

[11] Alexandru Emil Stanculescu. Bifibrations and Weak Factorisation Systems. *Applied Categorical Structures*, 20(1):19–30, 2012.

[12] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics.* `http://homotopytypetheory.org/book`, Institute for Advanced Study, 2013.