Compositional Z: Confluence Proofs for Permutative Conversion

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This paper gives new confluence proofs for several lambda calculi with permutation-like reduction, including lambda calculi corresponding to intuitionistic and classical natural deduction with disjunction and permutative conversions, and a lambda calculus with explicit substitution. In lambda calculi with permutative conversion, naïve parallel reduction techinique does not work, and (if we consider untyped terms, and hence we do not use strong normalization) traditional notion of residuals is required as Ando pointed out. This paper shows that the difficulties can be avoided by extending the technique proposed by Dehornoy and van Oostrom, called the Z theorem: existence of a mapping on terms with the Z property concludes the confluence. Since it is hard to directly define a mapping with the Z property for the lambda calculi with permutative conversions, this paper extends the Z theorem to compositional functions, called compositional Z, and shows that we can adopt it to the calculi.

1 Introduction

The permutative conversion was introduced by Prawitz [7] as one of proof normalization processes for the natural deduction with disjunctions and existential quantifiers. It permutes order of applications of elimination rules, and then normal proofs have some nice properties such as the subformula property. The rules of the permutative conversion are quite simple, but the combination of it with the β-reduction makes confluence proofs complicated if we do not depend on strong normalization as Ando discussed in [2]. First, we cannot naïvely adopt the parallel reduction technique of Tait and Martin-Löf, since a parallel reduction defined in an ordinary way does not have the diamond property. Therefore, Ando generalized the parallel reduction with the notion of the segment trees. Secondly, it is also difficult to adopt Takahashi's technique with complete development [8], and Ando used traditional notion of the residuals [3] to define the complete development.

This paper shows that we can avoid these troubles by adapting another proof technique for confluence proposed by Dehornoy and van Oostrom [4], called the *Z theorem*: if there is a mapping which satisfies the *Z property*, then the reduction system is confluent. A major candidate for the mapping with the Z property is the complete development used in Takahashi's proof, and hence defining such a mapping is still hard. In this paper, we see that a mapping satisfying the Z property can be easily defined as a composition of two complete developments with respect to the β-reduction and the permutative conversion, respectively, and the Z theorem is extended for compositional functions, called the *compositional Z*. The compositional Z can be adopted to several λ-calculi with permutative conversion such as the λ-calculus extended with disjunctions, the $\lambda\mu$-calculus with disjunctions, and a simple λ-calculus with explicit substitutions.

2 Compositional Z

First, we summarize Dehornoy and van Oostrom's Z theorem, and then extending it for compositional functions, called the compositional Z. It

Z 定理の拡張による置換簡約を含むラムダ計算の合流性証明

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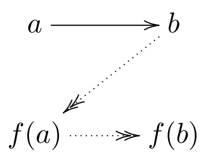
gives a sufficient condition for that a compositional function satisfies the Z property, and it enables us to consider a reduction system by dividing into two parts to prove confluence.

Definition 2.1 ((Weak) Z property)**.** Let $(A, \to)$ be an abstract rewriting system, and $\twoheadrightarrow$ be the reflexive transitive closure of $\to$. Let $\to_{\mathsf{x}}$ be another relation on A, and $\twoheadrightarrow_{\mathsf{x}}$ be its reflexive transitive closure.

1. A mapping f satisfies the *weak Z property for* $\to$ *by* $\to_{\mathsf{x}}$ if $a \to b$ implies $b \twoheadrightarrow_{\mathsf{x}} f(a) \twoheadrightarrow_{\mathsf{x}} f(b)$ for any $a, b \in A$.
2. A mapping f satisfies the *Z property for* $\to$ if it satisfies the weak Z property by $\to$ itself.

When f satisfies the (weak) Z property, we also say that f *is (weakly) Z*.

It becomes clear why we call it the Z property, when the condition is indicated by the following diagram.



Theorem 2.2 (Z theorem [4] [5])**.** *If there exists a mapping satisfying the Z property for an abstract rewriting system* $(A, \to)$*, then* $(A, \to)$ *is confluent.*

The following is easy to see by the diagram in Figure 1 and Theorem 2.2.

Theorem 2.3 (Compositional Z)**.** *Let* $(A, \to)$ *be an abstract rewriting system, and* $\to$ *be* $\to_1 \cup \to_2$*. If there exist mappings* $f_1, f_2 : A \to A$ *such that*

(a) f_1 *is Z for* $\to_1$

(b) $a \to_1 b$ *implies* $f_2(a) \twoheadrightarrow f_2(b)$

(c) $a \twoheadrightarrow f_2(a)$ *holds for any* $a \in \mathrm{Im}(f_1)$

(d) $f_2 \circ f_1$ *is weakly Z for* $\to_2$ *by* $\to$*,*

then $f_2 \circ f_1$ *is Z for* $(A, \to)$*, and hence* $(A, \to)$ *is confluent.*

Note that the condition (d) is weaker than the Z property of $f_2 \circ f_1$ for $(A, \to)$ since we only have to consider the one-step $\to_2$.

One easy example of the compositional Z is the $\beta\eta$-reduction on the untyped λ-calculus (although it can be directly proved by the Z theorem as in [5]). Let $\to_1 \; = \; \to_\eta$, $\to_2 \; = \; \to_\beta$, and f_1 and f_2 be the usual complete developments of η and β, respectively. Then, it is easy to see the conditions of Theorem 2.3 hold. The point is that we can forget the other reduction in the definition of each complete development.

Furthermore, we have another sufficient condition for the Z property of compositional functions as follows. It is a special case of Theorem 2.3, where $f_1(a) = f_1(b)$ holds for any $a \to_1 b$, and all of the examples in this paper (except for $\beta\eta$ above) of the application of compositional Z are in this case.

Corollary 2.4. *Let* $(A, \to)$ *be an abstract rewriting system, and* $\to$ *be* $\to_1 \cup \to_2$*. Suppose that there exist mappings* $f_1, f_2 : A \to A$ *such that*

(a) $a \to_1 b$ *implies* $f_1(a) = f_1(b)$

(b) $a \twoheadrightarrow_1 f_1(a)$ *for any* a

(c) $a \twoheadrightarrow f_2(a)$ *holds for any* $a \in \mathrm{Im}(f_1)$

(d) $f_2 \circ f_1$ *is weakly Z for* $\to_2$ *by* $\to$*.*

Then, $f_2 \circ f_1$ *is Z for* $(A, \to)$*, and hence* $(A, \to)$ *is confluent.*

Proof. It is easy to see from Theorem 2.3. The condition (a) in Theorem 2.3 comes from the new conditions (a) and (b), and (b) in Theorem 2.3 is not necessary since we have $f_2(f_1(a)) = f_2(f_1(b))$ for any $a \to_1 b$. □

This can be seen as a generalization of the *Z property modulo*, proposed by Accattoli and Kesner [1]. It concludes confluence of reduction system on quotient set $A/\sim$ for an equivalence relation $\sim$ by finding mapping which is well-defined on $\sim$ and is weakly Z for the reduction on A by the reduction relation modulo $\sim$. If we consider $\sim$ as the first reduction relation $\to_1$, and define $f_1(a)$ as a fixed representative of the equivalence class including a, then $\to$ means the reduction relation $\to_2$ modulo $\sim$, and Corollary 2.4 says the same thing as the Z property modulo.

3 Intuitionistic natural deduction with disjunction

3.1 Calculus

The following is the definition of the (untyped) terms (denoted by M, N,…), eliminators (denoted by e,…), and the reduction rules for the first-order natural deduction, where a ranges over the first-order variables, and t over the first-order terms. We call the system λ_{NJ}.

$$M ::= x \mid \lambda x.M \mid \langle M, M\rangle \mid \iota_1 M \mid \iota_2 M$$
$$\mid \lambda \mathsf{a}.M \mid \langle M, \mathsf{t}\rangle \mid Me$$
$$e ::= M \mid \pi_1 \mid \pi_2 \mid [x.M, x.M] \mid \mathsf{t} \mid [x\mathsf{a}.M]$$

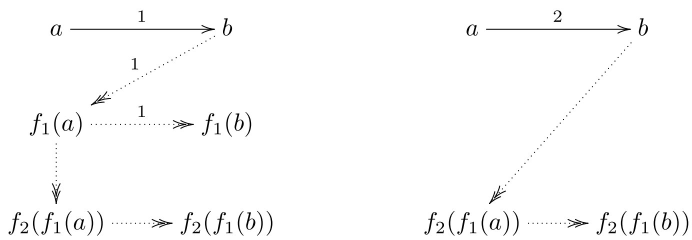


Fig. 1 Proof of Theorem 2.3

$$(\lambda x.M)N \to M[x := N] \quad (\beta_{\mathsf{I}})$$
$$\langle M_1, M_2\rangle\pi_i \to M_i \quad (\beta_{\mathsf{C}})$$
$$(\iota_i M)[x_1.N_1, x_2.N_2] \to N_i[x_i := M] \quad (\beta_{\mathsf{D}})$$
$$(\lambda \mathsf{a}.M)\mathsf{t} \to M[\mathsf{a} := \mathsf{t}] \quad (\beta_{\mathsf{A}})$$
$$\langle M, \mathsf{t}\rangle[x\mathsf{a}.N] \to N[x, \mathsf{a} := M, \mathsf{t}] \quad (\beta_{\mathsf{E}})$$
$$M[x_1.N_1, x_2.N_2]e \to M[x_1.N_1e, x_2.N_2e] \quad (\pi_{\mathsf{D}})$$
$$M[x\mathsf{a}.N]e \to M[x\mathsf{a}.Ne] \quad (\pi_{\mathsf{E}})$$

As usual, the juxtaposition notation Me is supposed to be left associative, and hence Me_1e_2 denotes $(Me_1)e_2$. The last two rules (π_{D}) and (π_{E}) are called *permutative conversion*, or just *permutation*. In these π-rules, we assume capture-avoiding conditions, that is, e must not contain either x_1 or x_2 freely in the rule (π_{D}) and e must not contain either x and a freely in the rule (π_{D}). In this paper, we consider untyped terms including ill-typed ones such as $(\iota_1 M)N$ and $(\lambda x.M)[x_1.N_1, x_2.N_2]$. They are just stuck since there is no applicable reduction rule.

In fact, we can discuss essence of our idea in the following simple subcalculus λ^-_{NJ}, which has proof terms only for implications and "unary" disjunctions. The discussion in this paper can be extended to λ_{NJ} in a straightforward way.

Definition 3.1 (λ^-_{NJ})**.** The terms of λ^-_{NJ} are defined as follows.

$$M ::= x \mid \lambda x.M \mid \iota M \mid Me \quad \text{(terms)}$$
$$e ::= M \mid [x.M] \quad \text{(eliminators)}$$

The reduction rules for λ^-_{NJ} are the following.

$$(\lambda x.M)N \to M[x := N] \quad (\beta_{\mathsf{I}})$$
$$(\iota M)[x.N] \to N[x := M] \quad (\beta_{\mathsf{D}})$$
$$M[x.N]e \to M[x.Ne] \quad (\pi)$$

The relation $\to$ on the terms is the compatible closure of these reduction rules, and $\twoheadrightarrow$ is its reflexive transitive closure.

3.2 Problems on confluence proof

In confluence proofs for λ^-_{NJ}, the permutation raises some difficulties, whichever we adopt the traditional parallel reduction or the original Z theorem (Theorem 2.2). A common reason can be explained by the example in Figure 2. The term M_3 is the least join point from M_1 and M_2. Hence, if we define a parallel reduction satisfying the diamond property, it has to contain $M_2 \twoheadrightarrow_\pi M_3$ as one-step, whereas in the reduction sequence

$$\begin{aligned} M_2 = M[x.N[y.L]]e &\to_\pi M[x.N[y.L]e] \\ &\to_\pi M[x.N[y.Le]] = M_3 \end{aligned}$$

the π-redex $N[y.L]e$ of the second step does not occur in M_2, and it is not a simple adaptation of the usual parallel reduction. This example also shows that, if we want to find a mapping f satisfying the Z property, we have to define $f(M_0)$ as $M_1 \twoheadrightarrow f(M_0)$ and $M_2 \twoheadrightarrow f(M_0)$ hold, and then $M_3 \twoheadrightarrow f(M_0)$. It means that we have to do the permutation completely in f. This observation leads the following definition.

Definition 3.2. The *complete permutation* $M@e$ is defined as follows.

$$M[x.N]@e = M[x.N@e]$$
$$M@e = Me \quad (M \neq M'[x.N'])$$

Then, we expect that a complete development with the complete permutation can be defined as follows and it is Z:

$$x^\bullet = x \qquad M^\bullet_{\mathsf{E}} = M^\bullet$$
$$(\lambda x.M)^\bullet = \lambda x.M^\bullet \qquad [x.N]^\bullet_{\mathsf{E}} = [x.N^\bullet].$$
$$(\iota M)^\bullet = \iota M^\bullet$$
$$(\lambda x.M)N^\bullet = M^\bullet[x := N^\bullet]$$
$$(\iota M)[x.N]^\bullet = N^\bullet[x := M^\bullet]$$
$$Me^\bullet = M^\bullet @ e^\bullet_{\mathsf{E}} \quad \text{(o.w.)}$$

However, this naïve definition does not work. Consider $N_1 = (\iota(x[y.y]))[z.z]w \to_\pi (\iota(x[y.y]))[z.zw] =$

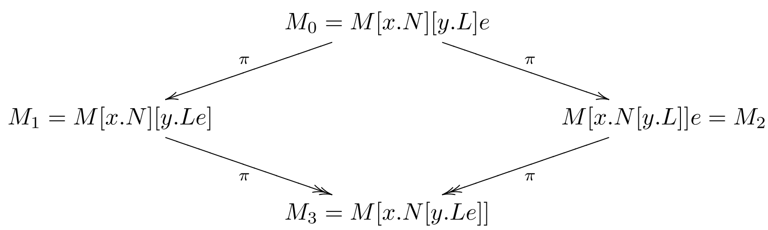


Fig. 2 Critical pair induced by the π-reduction

N_2. Then, we have

$$N_1{}^{\bullet} = (x[y.y])@w = x[y.yw]$$
$$N_2{}^{\bullet} = (zw)^{\bullet}[z := x[y.y]] = x[y.y]w,$$

and hence this mapping is not Z since $N_1^{\bullet} \twoheadrightarrow N_2{}^{\bullet}$ does not hold. The reason of the failure is that the π-redex $(x[y.y])w$ produced by the β-reduction is also reduced in $N_1{}^{\bullet}$.

3. 3 Confluence by compositional Z

The compositional Z can be used to solve the problem.

Definition 3.3. The mappings M^{P} and $e^{\mathsf{P}}_{\mathsf{E}}$ are inductively defined as follows.

$$x^{\mathsf{P}} = x \qquad M^{\mathsf{P}}_{\mathsf{E}} = M^{\mathsf{P}}$$
$$(\lambda x.M)^{\mathsf{P}} = \lambda x.M^{\mathsf{P}} \qquad [x.N]^{\mathsf{P}}_{\mathsf{E}} = [x.N^{\mathsf{P}}]$$
$$(\iota M)^{\mathsf{P}} = \iota M^{\mathsf{P}}$$
$$(Me)^{\mathsf{P}} = M^{\mathsf{P}}@e^{\mathsf{P}}_{\mathsf{E}}$$

The mappings M^{B} and $e^{\mathsf{B}}_{\mathsf{E}}$ are defined as follows.

$$x^{\mathsf{B}} = x \qquad M^{\mathsf{B}}_{\mathsf{E}} = M^{\mathsf{B}}$$
$$(\lambda x.M)^{\mathsf{B}} = \lambda x.M^{\mathsf{B}} \qquad [x.N]^{\mathsf{B}}_{\mathsf{E}} = [x.N^{\mathsf{B}}]$$
$$(\iota M)^{\mathsf{B}} = \iota M^{\mathsf{B}}$$
$$((\lambda x.M)N)^{\mathsf{B}} = M^{\mathsf{B}}[x := N^{\mathsf{B}}]$$
$$((\iota M)[x.N])^{\mathsf{B}} = N^{\mathsf{B}}[x := M^{\mathsf{B}}]$$
$$(Me)^{\mathsf{B}} = M^{\mathsf{B}}e^{\mathsf{B}}_{\mathsf{E}} \quad \text{(o.w.)}$$

We define $M^{\mathsf{PB}} = (M^{\mathsf{P}})^{\mathsf{B}}$.

Then, we can use Corollary 2.4 to show the confluence of $\lambda^{-}_{\mathsf{NJ}}$ with the help of the following lemmas.

Lemma 3.4. *1.* $Me \twoheadrightarrow_{\pi} M@e$*.*

2. $M@[x.N]@e = M@[x.N@e]$*.*

3. $(M@e)[x := N] \twoheadrightarrow_{\pi} M[x := N]@e[x := N]$*.*

4. $M \to M'$ *implies* $M@e \twoheadrightarrow M'@e$*.*

5. $e \to e'$ *implies* $M@e \twoheadrightarrow M@e'$*.*

Proof. 1. By induction on M. The only nontrivial case is the following, where $M = P[x.Q]$.

$$\begin{aligned} P[x.Q]e &\to_{\pi} P[x.Qe] \\ &\twoheadrightarrow_{\pi} P[x.Q@e] \qquad \text{(I.H.).} \end{aligned}$$

2. By induction on M. The only nontrivial case is the following, where $M = P[y.Q]$.

$$\begin{aligned} (P[y.Q])@[x.N]@e &= P[y.Q@[x.N]@e] \\ &= P[y.Q@[x.N@e]] \qquad \text{(I.H.)} \\ &= (P[y.Q])@[x.N@e]. \end{aligned}$$

3. By induction on M. We use θ to denote the substitution $[x := N]$. Interesting cases are the following.

(Case $M = P[y.Q]$) We have the following.

$$\begin{aligned} (M@e)\theta &= (P\theta)[y.(Q@e)\theta] \\ &\twoheadrightarrow_{\pi} (P\theta)[y.Q\theta@e\theta] \qquad \text{(I.H.)} \\ &= (P\theta[y.Q\theta])@e\theta \\ &= M\theta@e\theta. \end{aligned}$$

(Case $M = x$ and $N = P[y.Q]$) We have the following.

$$\begin{aligned} (M@e)\theta &= (xe)\theta \\ &= P[y.Q]e\theta \\ &\to_{\pi} P[y.Q(e\theta)] \\ &\twoheadrightarrow_{\pi} P[y.Q@e\theta] \qquad (1) \\ &= x\theta@e\theta. \end{aligned}$$

4. By induction on $M \to M'$. The only nontrivial cases are the following.

(Case $(\iota P)[x.Q] \to Q[x := P]$) We suppose that x does not occur in e.

$$\begin{aligned} (\iota P)[x.Q]@e &= (\iota P)[x.Q@e] \\ &\to (Q@e)[x := P] \\ &\twoheadrightarrow_{\pi} Q[x := P]@e \qquad (3). \end{aligned}$$

(Case $P[x.Q][y.R] \to P[x.Q[y.R]]$)

$$\begin{aligned} P[x.Q][y.R]@e &= P[x.Q][y.R@e] \\ &\to_{\pi} P[x.Q[y.R@e]] \\ &= P[x.Q[y.R]]@e. \end{aligned}$$

5. By induction on $e \to e'$. □

Lemma 3.5. $M \to_{\pi} N$ *implies* $M^{\mathsf{P}} = N^{\mathsf{P}}$*.*

Proof. By induction on $\to_{\pi}$. In the case of π-redex, we have the following.

$$\begin{aligned} (P[x.Q]e)^{\mathsf{P}} &= P^{\mathsf{P}}@[x.Q^{\mathsf{P}}]@e^{\mathsf{P}}_{\mathsf{E}} \\ &= P^{\mathsf{P}}@[x.Q^{\mathsf{P}}@e^{\mathsf{P}}_{\mathsf{E}}] \qquad (3.4.2) \\ &= (P[x.Qe])^{\mathsf{P}}. \end{aligned}$$

In the case of $M = Pe$ and $N = P'e'$, we have the following.

$$\begin{aligned}(Pe)^{\mathsf{P}} &= P^{\mathsf{P}}@e_{\mathsf{E}}^{\mathsf{P}} \\ &= P'^{\mathsf{P}}@e'^{\mathsf{P}}_{\mathsf{E}} && \text{(I.H.)} \\ &= P'@e'^{\mathsf{P}}.\end{aligned}$$

□

Lemma 3.6. *The following hold for* $\langle \mathsf{X}, \xi\rangle \in \{\langle \mathsf{P}, \pi\rangle, \langle \mathsf{B}, \beta\rangle\}$.

1. $M^{\mathsf{X}}[x := N^{\mathsf{X}}] \twoheadrightarrow_{\xi} (M[x := N])^{\mathsf{X}}$.

2. $M^{\mathsf{X}} e_{\mathsf{E}}^{\mathsf{X}} \twoheadrightarrow_{\xi} (Me)^{\mathsf{X}}$.

Proof. 1. By induction and case analysis on M. The only nontrivial cases are those where some redexes are created by substitutions.

($\mathsf{X} = \mathsf{P}$) The case where $M = xe$ and $N = P[y.Q]$ is proved as follows.

$$\begin{aligned}M^{\mathsf{P}}[x := N^{\mathsf{P}}] &= (P^{\mathsf{P}}@[y.Q^{\mathsf{P}}])e_{\mathsf{E}}^{\mathsf{P}}[x := N^{\mathsf{P}}] \\ &\twoheadrightarrow_{\pi} (P^{\mathsf{P}}@[y.Q^{\mathsf{P}}])(e[x := N])_{\mathsf{E}}^{\mathsf{P}} && \text{(I.H.)} \\ &\twoheadrightarrow_{\pi} P^{\mathsf{P}}@[y.Q^{\mathsf{P}}]@(e[x := N])_{\mathsf{E}}^{\mathsf{P}} && \text{(3.4.1)} \\ &= RHS\end{aligned}$$

($\mathsf{X} = \mathsf{B}$) The case where $M = xP$ and $N = \lambda y.Q$ is proved as follows.

$$\begin{aligned}M^{\mathsf{B}}[x := N^{\mathsf{B}}] &= (\lambda y.Q^{\mathsf{B}})P^{\mathsf{B}}[x := N^{\mathsf{B}}] \\ &\twoheadrightarrow_{\beta} (\lambda y.Q^{\mathsf{B}})(P[x := N])^{\mathsf{B}} && \text{(I.H.)} \\ &\rightarrow_{\beta} Q^{\mathsf{B}}[y := (P[x := N])^{\mathsf{B}}].\end{aligned}$$

The case where $M = x[y.P]$ and $N = \iota Q$ is proved as follows.

$$\begin{aligned}M^{\mathsf{B}}[x := N^{\mathsf{B}}] &= (\iota Q^{\mathsf{B}})[y.P^{\mathsf{B}}[x := N^{\mathsf{B}}]] \\ &\twoheadrightarrow_{\beta} (\iota Q^{\mathsf{B}})[y.(P[x := N])^{\mathsf{B}}] && \text{(I.H.)} \\ &\rightarrow_{\beta} (P[x := N])^{\mathsf{B}}[y := Q^{\mathsf{B}}].\end{aligned}$$

2. We have $M^{\mathsf{X}} e_{\mathsf{E}}^{\mathsf{X}} = (xe_{\mathsf{E}}^{\mathsf{X}})[x := M^{\mathsf{X}}] = (xe)^{\mathsf{X}}[x := M^{\mathsf{X}}] \twoheadrightarrow_{\xi} (Me)^{\mathsf{X}}$ by 1. □

Lemma 3.7. *For any* $\mathsf{X} \in \{\mathsf{P}, \mathsf{B}\}$*, if* $M \rightarrow N$ *holds, then we have* $M^{\mathsf{X}} \twoheadrightarrow N^{\mathsf{X}}$.

Proof. ($\mathsf{X} = \mathsf{P}$) The case where $M \rightarrow_{\pi} N$ immediately follows from Lemma 3.5. The case where M is a β-redex is proved as follows.

$$\begin{aligned}((\lambda x.P)Q)^{\mathsf{P}} &= (\lambda x.P^{\mathsf{P}})Q^{\mathsf{P}} \\ &= P^{\mathsf{P}}[x := Q^{\mathsf{P}}] \\ &\twoheadrightarrow_{\pi} (P[x := Q])^{\mathsf{P}} && \text{(3.6.1).}\end{aligned}$$

The case of $Me \rightarrow M'e'$ is proved by 4 and 5 of Lemma 3.4.

($\mathsf{X} = \mathsf{B}$) The only nontrivial case is the following: $M = (\iota P)[x.Q][y.R]$, $N = (\iota P)[x.Q[y.R]]$, and $M \rightarrow_{\pi} N$. In this case, we have $M^{\mathsf{B}} = Q^{\mathsf{B}}[x := P^{\mathsf{B}}][y.R^{\mathsf{B}}] = (Q^{\mathsf{B}}[y.R^{\mathsf{B}}])[x := P^{\mathsf{B}}]$ since x does not occur freely in $[y.R^{\mathsf{B}}]$, and $N^{\mathsf{B}} = (Q[y.R])^{\mathsf{B}}[x := P^{\mathsf{B}}]$. Therefore, $M^{\mathsf{B}} \twoheadrightarrow_{\beta} N^{\mathsf{B}}$ holds by Lemma 3.6. □

Theorem 3.8 (Confluence of $\lambda_{\mathsf{NJ}}^{-}$)**.** $\lambda_{\mathsf{NJ}}^{-}$ *is confluent.*

Proof. By Corollary 2.4, it is sufficient to prove the following.

(a) $M \rightarrow_{\pi} N$ implies $M^{\mathsf{P}} = N^{\mathsf{P}}$.

(b) $M \twoheadrightarrow M^{\mathsf{P}}$ holds for any M.

(c) $M \twoheadrightarrow M^{\mathsf{B}}$ holds for any M.

(d) $M \rightarrow_{\beta} N$ implies $N \twoheadrightarrow M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$.

(a) is Lemma 3.5. (b) and (c) are straightforward by induction on M. (d) $M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$ follows from Lemma 3.7. For $N \twoheadrightarrow M^{\mathsf{PB}}$, it is proved by induction on $M \rightarrow_{\beta} N$. The cases of β-redexes are easy by the fact $M \twoheadrightarrow M^{\mathsf{PB}}$ for any M, which follows from (b) and (c). The case where $M = Pe$ and $N = P'e'$ is proved as follows.

$$\begin{aligned}P'e' &\twoheadrightarrow P'^{\mathsf{PB}} e'^{\mathsf{PB}}_{\mathsf{E}} && \text{(I.H.)} \\ &\twoheadrightarrow (P'^{\mathsf{P}} e'^{\mathsf{P}}_{\mathsf{E}})^{\mathsf{B}} && \text{(3.6.2)} \\ &\twoheadrightarrow (P'^{\mathsf{P}}@e'^{\mathsf{P}}_{\mathsf{E}})^{\mathsf{B}} && \text{(3.4.1, 3.7).}\end{aligned}$$

□

4 Classical natural deduction with disjunction

The idea in the previous section can be extended to the Parigot's $\lambda\mu$-calculus [6]. As Ando's proof in [2], the proof of its confluence requires some complicated notions such as generalized parallel reduction, which is realized by means of the notion of segment trees, and residuals of redexes to define the complete development. The compositional Z makes the proof much simpler.

Definition 4.1. The terms of $\lambda\mu_{\mathsf{NK}}^{-}$ are the extension of those of $\lambda_{\mathsf{NJ}}^{-}$ as follows.

$$M ::= \cdots \mid \mu\alpha.M \mid [\alpha]M \qquad \text{(terms)}$$

The following is the additional reduction rule.

$$(\mu\alpha.M)e \rightarrow \mu\alpha.M[\alpha \Leftarrow e] \qquad (\mu),$$

where the structural substitution $M[\alpha \Leftarrow e]$ is obtained by recursively replacing subterms of the form $[\alpha]M$ by $[\alpha]Me$.

In the following, we use the notation $M[\alpha \Leftarrow e_1, e_2, \cdots, e_n]$ to denote $M[\alpha \Leftarrow e_1][\alpha \Leftarrow e_2] \cdots [\alpha \Leftarrow e_n]$. This term is obtained by replacing $[\alpha]N$ by $[\alpha]Ne_1e_2 \cdots e_n$.

Extending the complete permutation to $\lambda\mu_{\mathsf{NK}}^{-}$

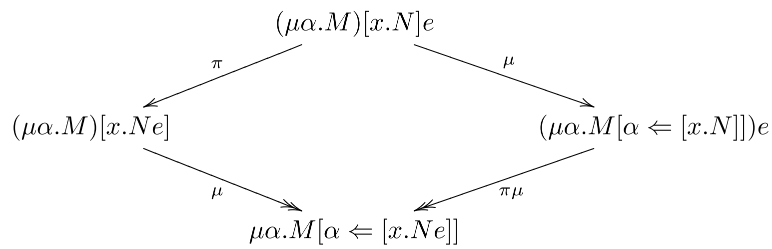


Fig. 3 Critical pair induced by the $\pi\mu$-reduction

is not straightforward. First, we have to do μ-reduction simultaneously in the complete permutation because of the example in Figure 3. Here, the right bottom arrow needs one μ-step followed by some π-steps as

$$\begin{aligned}(\mu\alpha.M[\alpha \Leftarrow [x.N]])e &\to_\mu \mu\alpha.M[\alpha \Leftarrow [x.N], e]\\ &\twoheadrightarrow_\pi \mu\alpha.M[\alpha \Leftarrow [x.Ne]].\end{aligned}$$

Note that the π-reduction in the second line holds by π-reducing subterms of the form $[\alpha]P[x.N]e$ in $M[\alpha \Leftarrow [x.N], e]$ to $[\alpha]P[x.Ne]$. Secondly, the following naïve definition is not inductive on the size of terms:

$$\begin{aligned}(\mu\alpha.M)@e &= \mu\alpha.M[\alpha \Leftarrow @e]\\ ([\alpha]M)[\alpha \Leftarrow @e] &= [\alpha](M[\alpha \Leftarrow @e]@e).\end{aligned}$$

Hence, we need some generalization for the definition of the complete permutation with respect to both π- and μ-reduction.

Definition 4.2. We use the following notation. The metavariable ε ranges over eliminators or $\circ$ denoting "nothing", and we define

$$M\varepsilon = \begin{cases} Me & (\varepsilon = e)\\ M & (\varepsilon = \circ).\end{cases}$$

$\overline{\alpha}$ and $\overline{\varepsilon}$ denote finite sequences such as $\alpha_1, \alpha_2, \cdots, \alpha_n$ and $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$, respectively, and $\bullet$ denotes the empty sequence.

We define $M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon$ as Figure 4, where we suppose that $\overline{\alpha}$ and $\overline{\varepsilon}$ have the same length.

Then, we define $M@e = M[\bullet \Leftarrow @\bullet]@e$ and $M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}] = M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ$.

Note that, the following equations hold as we expect.

$$\begin{aligned}(M[x.N])@e &= M[x.N@e]\\ (\mu\alpha.M)@e &= \mu\alpha.M[\alpha \Leftarrow @e]\\ M@e &= Me \qquad \text{(o.w.)}\end{aligned}$$

Now, we can define a function with the Z property by composing two functions in a similar way to the case of λ^-_{NJ}.

Definition 4.3. The mappings M^{P} and $e^{\mathsf{P}}_{\mathsf{E}}$ are inductively defined as follows.

$$\begin{aligned}x^{\mathsf{P}} &= x & M^{\mathsf{P}}_{\mathsf{E}} &= M^{\mathsf{P}}\\ (\lambda x.M)^{\mathsf{P}} &= \lambda x.M^{\mathsf{P}} & [x.N]^{\mathsf{P}}_{\mathsf{E}} &= [x.N^{\mathsf{P}}]\\ (\iota M)^{\mathsf{P}} &= \iota M^{\mathsf{P}}\\ ([\alpha]M)^{\mathsf{P}} &= [\alpha]M^{\mathsf{P}}\\ (\mu\alpha.M)^{\mathsf{P}} &= \mu\alpha.M^{\mathsf{P}}\\ (Me)^{\mathsf{P}} &= M^{\mathsf{P}}@e^{\mathsf{P}}_{\mathsf{E}}\end{aligned}$$

The mappings M^{B} and $e^{\mathsf{B}}_{\mathsf{E}}$ are defined as follows.

$$\begin{aligned}x^{\mathsf{B}} &= x & M^{\mathsf{B}}_{\mathsf{E}} &= M^{\mathsf{B}}\\ (\lambda x.M)^{\mathsf{B}} &= \lambda x.M^{\mathsf{B}} & [x.N]^{\mathsf{B}}_{\mathsf{E}} &= [x.N^{\mathsf{B}}]\\ (\iota M)^{\mathsf{B}} &= \iota M^{\mathsf{B}}\\ ([\alpha]M)^{\mathsf{B}} &= [\alpha]M^{\mathsf{B}}\\ (\mu\alpha.M)^{\mathsf{B}} &= \mu\alpha.M^{\mathsf{B}}\\ ((\lambda x.M)N)^{\mathsf{B}} &= M^{\mathsf{B}}[x := N^{\mathsf{B}}]\\ ((\iota M)[x.N])^{\mathsf{B}} &= N^{\mathsf{B}}[x := M^{\mathsf{B}}]\\ (Me)^{\mathsf{B}} &= M^{\mathsf{B}}e^{\mathsf{B}}_{\mathsf{E}} \quad \text{(o.w.)}\end{aligned}$$

We define $M^{\mathsf{PB}} = (M^{\mathsf{P}})^{\mathsf{B}}$.

Then, we can use Theorem 2.3 to show the confluence of $\lambda\mu^-_{\mathsf{NK}}$ with the help of several lemmas.

Theorem 4.4 (Confluence of $\lambda\mu^-_{\mathsf{NK}}$)**.** $\lambda\mu^-_{\mathsf{NK}}$ *is confluent.*

Proof. By Corollary 2.4, it is sufficient to prove the following.

(a) $M \to_{\pi\mu} N$ implies $M^{\mathsf{P}} = N^{\mathsf{P}}$
(b) $M \twoheadrightarrow M^{\mathsf{P}}$ holds for any M.
(c) $M \twoheadrightarrow M^{\mathsf{B}}$ holds for any M.
(d) $M \to_\beta N$ implies $N \twoheadrightarrow M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$.

These are proved in a similar way to the case of λ^-_{NJ}. □

5 Explicit substitutions

As another example of an application of the compositional Z, we show confluence of the simplest calculus with explicit substitutions, in which the propagation rules look like the permutation rules.

Definition 5.1 (λ_{x})**.** Terms of λ_{x} are defined as

$$
\begin{aligned}
x[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= x\varepsilon \\
(\lambda x.M)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= (\lambda x.M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)\varepsilon \\
(\iota M)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= (\iota M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)\varepsilon \\
(MN)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= (M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)(N[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)\varepsilon \\
(M[x.N])[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= (M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)[x.N[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon] \\
(\mu\beta.M)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= \mu\beta.M[\overline{\alpha}, \beta \Leftarrow @\overline{\varepsilon}, \varepsilon]@\circ \\
([\alpha_i]M)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= ([\alpha_i]M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon_i)\varepsilon && (\alpha_i \in \overline{\alpha}) \\
([\beta]M)[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon &= ([\beta]M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\circ)\varepsilon && (\beta \notin \overline{\alpha})
\end{aligned}
$$

Fig. 4 Definition of $M[\overline{\alpha} \Leftarrow @\overline{\varepsilon}]@\varepsilon$

follows.

$$M ::= x \mid \lambda x.M \mid MM \mid M\langle x := M\rangle$$

In the term $M\langle x := N\rangle$, the variable occurrences of x in M are bound, and it is supposed that we can freely rename bound variables as usual.

Reduction rules of λ_{x} are the following, where x and y are distinct variables, and, in the rule (π_{abs}), x does not occur freely in N.

$$
\begin{aligned}
(\lambda x.M)N &\to M\langle x := N\rangle && (\beta_{\mathsf{x}}) \\
y\langle y := N\rangle &\to N && (\pi_{\mathsf{hit}}) \\
x\langle y := N\rangle &\to N && (\pi_{\mathsf{gc}}) \\
(\lambda x.P)\langle y := N\rangle &\to \lambda x.P\langle y := N\rangle && (\pi_{\mathsf{abs}}) \\
(PQ)\langle y := N\rangle &\to P\langle y := N\rangle Q\langle y := N\rangle && (\pi_{\mathsf{app}})
\end{aligned}
$$

The outline of the following proof with the compositional Z is almost the same as the case of $\lambda^{-}_{\mathsf{NJ}}$ and $\lambda\mu^{-}_{\mathsf{NK}}$. In this case, what corresponds complete permutation $(\cdot)^{\mathsf{P}}$ is to change explicit substitutions $\langle x := M\rangle$ to meta substitutions $[x := M]$.

Definition 5.2. The mappings M^{P} and M^{B} are defined as follows.

$$
\begin{aligned}
x^{\mathsf{P}} &= x \\
(\lambda x.M)^{\mathsf{P}} &= \lambda x.M^{\mathsf{P}} \\
(MN)^{\mathsf{P}} &= M^{\mathsf{P}}N^{\mathsf{P}} \\
(M\langle x := N\rangle)^{\mathsf{P}} &= M^{\mathsf{P}}[x := N^{\mathsf{P}}]
\end{aligned}
$$

$$
\begin{aligned}
x^{\mathsf{B}} &= x \\
(\lambda x.M)^{\mathsf{B}} &= \lambda x.M^{\mathsf{B}} \\
((\lambda x.M)N)^{\mathsf{B}} &= M^{\mathsf{B}}[x := N^{\mathsf{B}}] \\
(MN)^{\mathsf{B}} &= M^{\mathsf{B}}N^{\mathsf{B}} && \text{(o.w.)} \\
(M\langle x := N\rangle)^{\mathsf{B}} &= M^{\mathsf{B}}[x := N^{\mathsf{B}}]
\end{aligned}
$$

Then, we define $M^{\mathsf{PB}} = (M^{\mathsf{P}})^{\mathsf{B}}$.

In fact, the last equation of the definition of $(\cdot)^{\mathsf{B}}$ is not used, because it is applied only to terms without explicit substitutions in the following discussion.

It is easy to see the following auxiliary lemmas.

Lemma 5.3. *1. $M \to_\pi N$ implies $M^{\mathsf{P}} = N^{\mathsf{P}}$.*

2. M^{P} contains no explicit substitution.

3. If M contains no explicit substitution, then we have $M^{\mathsf{P}} = M$.

4. If M contains no explicit substitution, then we have $M\langle x := N\rangle \twoheadrightarrow_\pi M[x := N]$.

Proof. 1 is proved by induction on $M \to_\pi N$, and 2, 3, and 4 are by induction on M. □

Lemma 5.4. *1. If $M \to N$ holds in λ_{x}, then we have $M^{\mathsf{P}} \twoheadrightarrow_\beta N^{\mathsf{P}}$ in the ordinary λ-calculus (without explicit substitutions).*

2. For M and N containing no explicit substitution, if $M \to_\beta N$ holds in the ordinary λ-calculus, then we have $M \twoheadrightarrow N$ in λ_{x}.

Proof. 1 is proved by induction on $M \to N$, and 2 is by induction on $M \to_\beta N$. □

On terms without explicit substitutions, the mapping $(\cdot)^{\mathsf{B}}$ is the ordinary complete development, and it has the Z property for the β-reduction in the λ-calculus [5].

Now we can prove confluence of λ_{x} by the compositional Z.

Theorem 5.5 (Confluence of λ_{x})**.** *λ_{x} is confluent.*

Proof. By Corollary 2.4, it is sufficient to prove the following.

(a) $M \to_\pi N$ implies $M^{\mathsf{P}} = N^{\mathsf{P}}$

(b) $M \twoheadrightarrow M^{\mathsf{P}}$ holds for any M.

(c) $M \twoheadrightarrow M^{\mathsf{B}}$ holds for any M without explicit substitutions.

(d) $M \to_{\beta_{\mathsf{x}}} N$ implies $N \twoheadrightarrow M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$.

(a) is Lemma 5.3.1. (b) is easy by Lemma 5.3.4.

(c) is also easy since we have

$$(\lambda x.P)Q \twoheadrightarrow_{\beta_\mathsf{x}} P\langle x := Q\rangle \twoheadrightarrow_\pi P[x := Q]$$

by Lemma 5.3.4. (d) is proved by induction on $M \to_{\beta_\mathsf{x}} N$. For $N \twoheadrightarrow M^{\mathsf{PB}}$, the only nontrivial case where $P\langle x := Q\rangle \to_{\beta_\mathsf{x}} P'\langle x := Q'\rangle$ is proved as follows.

$$\begin{aligned} P'\langle x := Q'\rangle &\twoheadrightarrow P^{\mathsf{PB}}\langle x := Q^{\mathsf{PB}}\rangle && \text{(I.H., (b), (c))} \\ &\twoheadrightarrow P^{\mathsf{PB}}[x := Q^{\mathsf{PB}}] && (5.3.4) \\ &\twoheadrightarrow (P^{\mathsf{P}}[x := Q^{\mathsf{P}}])^{\mathsf{B}}, \end{aligned}$$

where, for the last line, we can prove $M^{\mathsf{B}}[x := N^{\mathsf{B}}] \twoheadrightarrow_\beta (M[x := N])^{\mathsf{B}}$ in the λ-calculus in a similar way to Lemma 3.6.1, and hence we have $M^{\mathsf{B}}[x := N^{\mathsf{B}}] \twoheadrightarrow (M[x := N])^{\mathsf{B}}$ in λ_x by Lemma 5.4.2. The rest part of (d), $M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$, is proved as follows. Suppose that $M \to_{\beta_\mathsf{x}} N$ holds, and we have $M^{\mathsf{P}} \twoheadrightarrow_\beta N^{\mathsf{P}}$ in the λ-calculus by Lemma 5.4.1. Then, $M^{\mathsf{PB}} \twoheadrightarrow_\beta N^{\mathsf{PB}}$ since $(\cdot)^{\mathsf{B}}$ is Z for β, and hence $M^{\mathsf{PB}} \twoheadrightarrow N^{\mathsf{PB}}$ in λ_x. □

6 Concluding remark

We have proposed an extension of Dehornoy and van Oostrom's Z theorem, called the compositional Z. This idea can be widely applied to lambda calculi with permutative conversions, including the λ- and the $\lambda\mu$-calculi with disjunction and permutative conversion, and a simple variant of lambda calculus with explicit substitutions, where the propagation of the explicit substitutions is similar to the permutation rules. In particular, the combination of the β-reduction and the permutative conversions makes the confluence proofs much difficult to define the parallel reduction or a mapping with the Z property. We have seen that the latter is easily defined as a compositional function, and hence the compositional Z gives simple confluence proofs for these calculi.

The compositional Z also gives a new possibility toward modular (or gradual) proofs of confluence. In general, it is hard to prove confluence by dividing a reduction system into some parts because of the non-modular character of confluence. The compositional Z enables us to reuse the Z property for a subsystem, that is, for $\to_1$, a subrelation of $\to$, the Z property for $\to_1$ can be used to prove the Z property for $\to$ by the compositional Z.

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